PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 1: Sample Space and Probability

STAT/MA 41600
Practice Problems: August 27, 2014
Solutions by Mark Daniel Ward

Problem Set 1 Answers

1. Choosing points at random. (a.) The sample space is

$$S = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2 - x\}.$$

(b.) The sample space is

$$S = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 4 - x\}.$$

2. Gloves. (a.) Any of the 5 gloves can come out first, followed by any of the 4 remaining gloves, followed by any of the 3 remaining gloves, followed by any of the 2 remaining gloves, followed by the last glove. So there are $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ possible outcomes.

(b.) Each outcome in part (b) represents exactly $2 \times 2 = 4$ of the outcomes from part (a). For instance, this outcome from part (b):

(“blue”, “white”, “red”, “red”, “blue”).

corresponds to these four outcomes from part (a):

(“blue right”, “white right”, “red left”, “red right”, “blue left”),

(“blue left, “white right”, “red left”, “red right”, “blue right),

(“blue right”, “white right”, “red right”, “red left”, “blue left”),

(“blue left, “white right”, “red right”, “red left”, “blue right).

So there are only 1/4 as many outcomes in part (b), as compared to part (a). Thus there are $\frac{120}{4} = 30$ outcomes possible in part (b).

3. Seating arrangements.

Method #1: Alice sits somewhere. Someone, say a “mystery” person, sits on Alice’s right-hand side. Regardless of who this mystery person is, the mystery person will be sitting next to her/his spouse in one of three possible ways (and of course the other couple will be together in such a case too). So the couples are happy in 1 out of every 3 possible arrangements, i.e., in $24/3 = 8$ of the possible arrangements.

Method #2: Alice sits somewhere. For the couples to sit together, either Bob and Catherine (collectively, as a pair) sit on Alice’s left or right. In either case, Bob and Catherine have two ways of sitting among themselves (i.e., within their own pair). Within the two remaining seats, Doug and Edna have two ways of sitting among themselves (i.e., within their own pair). So there are $2 \times 2 \times 2 = 8$ possible ways that the couples can sit happily.

4. Abstract art.

An event is completely known once we decide whether or not each outcome belongs in the event. So, for each of the five outcomes, we just decide ("yes" or "no") whether that outcome goes in the event we are building. So there are $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$ possible events that we could build.

5. Sum of three dice. We have:

event	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
number of outcomes	1	3	6	10	15	21	25	27

event	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}
number of outcomes	27	25	21	15	10	6	3	1

Perhaps the easiest way to solve this problem is to go through the possible choices of values for the first die, and then use the chart suggested, to see how many outcomes are possible for the second and third die.

For instance, to determine A_{15}, we could have:
If the first die is 1 or 2, the second and third dice cannot be large enough for a sum of 15.
If the first die is 3, there is 1 way the second and third dice add up to 12.
If the first die is 4, there are 2 ways the second and third dice add up to 11.
If the first die is 5, there are 3 ways the second and third dice add up to 10.
If the first die is 6, there are 4 ways the second and third dice add up to 9.

So A_{15} has $1 + 2 + 3 + 4 = 10$ possible outcomes.

The problem is completely symmetric too, so once you have done A_{11} through A_{18}, the other counts of outcomes are the same, in reverse order.

STAT/MA 41600
Practice Problems: August 29, 2014
Solutions by Mark Daniel Ward

1. Choosing a page at random.
Let A_1 be the event that the ones digit is a "5"; let A_2 be the event that the tens digit is a "5"; let A_3 be the event that the hundreds digit is a "5". Then we want to find

$$P(A_1 \cup A_2 \cup A_3).$$

By inclusion-exclusion, we have

$$\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\
&\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} + \frac{1}{1000} \\
&= 271/1000.
\end{aligned}$$

2. Gloves. (a.) For A_j to occur, he must miss the white glove exactly $j-1$ times, and then finally get the white glove. So $P(A_j) = \left(\frac{4}{5}\right)^{j-1}\left(\frac{1}{5}\right)$.

(b.) *Method #1:*

$$\begin{aligned}
P(B_1) &= \frac{1}{5} \\
P(B_2) &= \frac{(4)(1)}{(5)(4)} = \frac{1}{5} \\
P(B_3) &= \frac{(4)(3)(1)}{(5)(4)(3)} = \frac{1}{5} \\
P(B_4) &= \frac{(4)(3)(2)(1)}{(5)(4)(3)(2)} = \frac{1}{5} \\
P(B_5) &= \frac{(4)(3)(2)(1)(1)}{(5)(4)(3)(2)(1)} = \frac{1}{5}
\end{aligned}$$

Method #2: We could just notice that, if he continues to pull all the gloves out of the drawer until it is empty, the white glove is equally likely to appear in any of the 5 positions. Thus $P(B_k) = 1/5$ for each k.

3. Seating arrangements. *Method #1:* Alice sits somewhere. Someone, say a "mystery" person, sits on Alice's right-hand side. If the mystery person's spouse sits beside him/her, then both couples are happy; this happens with probability 1/3, so $P(A_2) = 1/3$. If the mystery person's spouse sits one seat away from him/her, then neither couple is happy; this happens with probability 1/3, so $P(A_0) = 1/3$. If the mystery person sits two seats away from him/her (i.e., all the way around the table, on Alice's other side), then only the other couple is happy; this happens with probability 1/3, so $P(A_1) = 1/3$.

Method #2: Alice sits somewhere. For the couples to sit together, either Bob and Catherine (collectively, as a pair) sit on Alice's left or right. In either case, Bob and Catherine have two ways of sitting among themselves (i.e., within their own pair). Within the two remaining seats, Doug and Edna have two ways of sitting among themselves (i.e., within their own pair). So there are $2 \times 2 \times 2 = 8$ possible ways that the couples can sit happily. So $P(A_2) = 8/24 = 1/3$. For neither couple to sit together, someone sits on Alice's left (4 choices), and then his/her spouse must be 2 chairs away (1 choice), and then there are two ways that the remaining couple can be seated in the two remaining chairs (2 choices). Thus $P(A_0) = 8/24 = 1/3$. For exactly one couple to sit together, someone sits on Alice's left (4 choices), and then his/her spouse must be on Alice's right (1 choice), and then there are two ways that the remaining couple can be seated together in the two remaining chairs (2 choices). Thus $P(A_1) = 8/24 = 1/3$.

4. Abstract art.

EVENT:	Probability of the event:
$\{(P)\}$	1/3, since Purple is first 1/3 of the time.
$\{(G,P),(Y,P)\}$	1/3, since Purple is second 1/3 of the time
$\{(G,P),(G,Y,P)\}$	1/3, since Green is first 1/3 of the time
$\{(Y,G,P),(G,Y,P)\}$	1/3, since Purple is last 1/3 of the time

$\{(P),(Y,P)\}$ $1/3 + 1/6 = 1/2$. The first probability is 1/3 since Purple is first 1/3 of the time. The second probability is 1/6 since there is only 1 out of $(3)(2) = 6$ ways to pick Yellow then Purple. The events in the two previous sentences are disjoint, so we can just add their probabilities.

5. Maximum of three dice. Event B_k occurs if and only if all three rolls are between 1 and k. So B_k occurs in $k \times k \times k = k^3$ ways out of the $6^3 = 216$ ways altogether. Thus $P(B_k) = \frac{k^3}{216} = (k/6)^3$.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 2: Independent Events, Conditional Probability and Bayes' Theorem

STAT/MA 41600
Practice Problems: September 3, 2014
Solutions by Mark Daniel Ward

1. Choosing a page at random. Yes, the events are independent. We compute

$$P(A) = 100/1000 = 1/10; \qquad P(B) = 10/1000 = 1/100; \qquad P(A \cap B) = 1/1000.$$

So $P(A)P(B) = P(A \cap B)$. So A, B are independent.

2. Gloves. Using the ideas from the notes, about finding a good result (here, a red glove) before a bad one (here, a white glove), we can write $p = 2/5$ and $q = 1/5$, and the desired probability is $\frac{p}{p+q} = \frac{2/5}{2/5+1/5} = 2/3$.

As a another (very intuitive) method of solution, just look at the first draw which is not blue. Such a draw is red with probability 2/3 (as desired), or white with probability 1/3.

If you prefer to grind out the answer, it might be easier to compute the complement, i.e., that the white glove arrives without ever having seen a red glove. For this to happen in exactly j draws has probability:

$$\overbrace{(2/5)(2/5)\cdots(2/5)}^{j-1}(1/5) = (2/5)^{j-1}(1/5).$$

So the probability of getting white before red appears is

$$\sum_{j=1}^{\infty}(2/5)^{j-1}(1/5) = (1/5)\sum_{j=0}^{\infty}(2/5)^{j} = \frac{1/5}{1-2/5} = 1/3.$$

Thus the probability of getting red before white (i.e., the probability of the complementary event) is 2/3, as we claimed above.

3. Seating arrangements. No, events T and U are not independent; rather, T, U are dependent. For any given pair of people, the probability of them sitting next to each other is $2/4 = 1/2$, because the first member of the pair can sit anywhere and the other member of the pair can sit next to her/him in two out of the four possible seats, all of which are equally likely. Thus $P(T) = 1/2$ and $P(U) = 1/2$.

On the other hand, to compute $P(T \cap U)$, we need Bob to sit by Alice and Catherine. Bob can sit anywhere, and then Alice always has 2 out of the 4 remaining possible places to sit beside Bob, and this leaves Catherine 1 out of the 3 remaining possible places to sit beside Bob. Thus $P(T \cap U) = (2/4)(1/3) = 1/6 \neq 1/4 = P(T)P(U)$. So T, U are dependent.

4. Abstract art. Yes, A, B are independent. First, $P(A) = 1/3$ because purple is equally likely to be found in any of the three possible trials. Also, $P(B) = 1/2$ because green is equally likely to be found before or after yellow. Finally, $P(A \cap B) = 1/6$ because the 6 outcomes are equally likely, and $A \cap B$ only occurs if (G, P, Y) is the outcome.

5. Even versus four or less. Yes, A, B are independent. We have

$$\begin{aligned} P(A) &= P(\{2,4,6\}) = 3/6 = 1/2; \\ P(B) &= P(\{1,2,3,4\}) = 4/6 = 2/3; \\ P(A \cap B) &= P(\{2,4\}) = 2/6 = 1/3 = P(A)P(B). \end{aligned}$$

No, B, C are not independent; these events are dependent. We have

$$\begin{aligned} P(B) &= P(\{1,2,3,4\}) = 4/6 = 2/3; \\ P(C) &= P(\{3,4,5,6\}) = 4/6 = 2/3; \\ P(B \cap C) &= P(\{3,4\}) = 2/6 = 1/3 \neq 4/9 = P(B)P(C). \end{aligned}$$

STAT/MA 41600
Practice Problems: September 5, 2014
Solutions by Mark Daniel Ward

1. Choosing a page at random. (a.) We have $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. Since $A \subset B$, then $P(A \cap B) = P(A) = 1/1000$. Also $P(B) = 271/1000$, as discovered on problem set 2. Thus $P(A \mid B) = \frac{1/1000}{271/1000} = 1/271$.

(b.) We have $P(A \mid C) = \frac{P(A \cap C)}{P(C)}$. Since $A \subset C$, then $P(A \cap C) = P(A) = 1/1000$.

We compute $P(C)$ using inclusion-exclusion, as in problem set 2. Let C_1 be the event that the ones and tens digits are "5"s; let C_2 be the event that the tens and hundreds digits are "5"s; let C_3 be the event that the one and hundreds digits are "5"s. Then we want $P(C) = P(C_1 \cup C_2 \cup C_3)$. By inclusion-exclusion, we have

$$\begin{aligned} P(C_1 \cup C_2 \cup C_3) &= P(C_1) + P(C_2) + P(C_3) \\ &\quad - P(C_1 \cap C_2) - P(C_1 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3) \\ &= \frac{1}{100} + \frac{1}{100} + \frac{1}{100} - \frac{1}{1000} - \frac{1}{1000} - \frac{1}{1000} + \frac{1}{1000} \\ &= 28/1000. \end{aligned}$$

Thus $P(A \mid C) = \frac{1/1000}{28/1000} = 1/28$.

2. Gloves. (a.) As before, $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. Also $A \subset B$, so $P(A \cap B) = P(A)$. The probability both gloves are blue is $P(A) = (\frac{2}{5})(\frac{1}{4}) = \frac{1}{10}$. The probability neither glove is blue is $(\frac{3}{5})(\frac{2}{4}) = \frac{3}{10}$; the complementary probability is $P(B) = 1 - 3/10 = 7/10$. Thus $P(A \mid B) = \frac{1/10}{7/10} = 1/7$.

(b.) Since conditional probabilities satisfy all of the three standard axioms of probability, then $P(A^c \mid B) = 1 - P(A \mid B) = 1 - 1/7 = 6/7$.

3. Seating arrangements. Given that Bob and Catherine are sitting next to each other, there are three remaining consecutive seats in a row. If Alice sits in the middle, then Doug and Edna are separated; this happens with probability 1/3. If Alice sits on either end of the three remaining seats, then Doug and Edna are together; this happens with probability 2/3. Thus $P(A \mid B) = 2/3$.

4. Pair of dice. Exactly 6 out of the 36 equally likely outcomes have the same result on both die (i.e., are "doubles"). Thus, $P(B) = 30/36$ is the probability that the values on the dice do not agree.

Also, exactly 18 out of 36 of the outcomes are even; exactly 6 of these outcomes are "doubles", and the other 12 have even sums but are not doubles. So $P(A \cap B) = 12/36$.

So we conclude $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{12/36}{30/36} = 12/30 = 2/5$.

5. Pair of dice. Exactly 10 out of the 36 equally likely outcomes have sum 9 or larger. Thus, $P(B) = 10/36$. Also $A \subset B$, so $P(A \cap B) = P(A)$. Also, exactly 3 out of the 36 equally likely outcomes have sum exactly 10, so $P(A) = 3/36$. Thus So we conclude $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{10/36} = 3/10$.

STAT/MA 41600
Practice Problems: September 8, 2014
Solutions by Mark Daniel Ward

1. Waking up at random. 1a. Writing A as the event it is a weekday, and B as the event it is before 8 AM, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(.65)(5/7)}{(.65)(5/7) + (.22)(2/7)} = .881.$$

1b. Writing A as the event it is a weekday, and B as the event it is after 8 AM, we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(.35)(5/7)}{(.35)(5/7) + (.78)(2/7)} = .529.$$

2. Gloves. We really need to know whether the first glove is white or blue. So let B be the event that the second glove is blue. Let A be the event that the first glove is blue or white. We want $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$. Of course $P(A) = 3/5$. Also $P(A \cap B) = (2/5)(1/4) + (1/5)(2/4) = 1/5$. So $P(B \mid A) = 1/3$.

3. Pair of dice. Let A be the event that the sum of the two dice is exactly 4. Let B be the event that the blue die has an odd value.

Then
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Notice $P(B) = 3/6 = 1/2$ since the blue die has an odd value exactly 1/2 the time. Also $P(A \cap B) = 2/36 = 1/18$ since only 2 out of the 36 equally likely outcomes are in $A \cap B$, namely, if the (blue,red) values are (1,3) or (3,1). Thus $P(A \mid B) = \frac{1/18}{1/2} = 1/9$.

4. Pair of dice. Let A be the event that the sum of the two dice is 7 or larger. Let B be the event that the blue die has a value of 4 or smaller. Then $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. Notice $P(B) = 4/6 = 2/3$.

It is easier to calculuate $P(A \cap B)$ if we break B up into four smaller events B_1, B_2, B_3, B_4, namely, the events that the blue die has value exactly 1, 2, 3, or 4, respectively. These are disjoint events, so

$$\begin{aligned}
P(A \cap B) &= P(A \cap (B_1 \cup B_2 \cup B_3 \cup B_4)) \\
&= P((A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4)) \\
&= P(A \cap B_1) + (A \cap B_2) + (A \cap B_3) + (A \cap B_4) \\
&= P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3) + P(A \mid B_4)P(B_4) \\
&= (1/6)(1/6) + (2/6)(1/6) + (3/6)(1/6) + (4/6)(1/6) = 10/36 = 5/18
\end{aligned}$$

Thus $P(A \mid B) = \frac{5/18}{2/3} = 5/12$.

5. Coin flips and then dice. Let B be the event that none of the dice show the value 1, and let A_k be the event that exactly k flips are needs to get heads. Then

$$\begin{aligned}
P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + P(A_4 \cap B) + \cdots \\
&= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) + P(B \mid A_4)P(A_4) + \cdots \\
&= (5/6)^1(1/2)^1 + (5/6)^2(1/2)^2 + (5/6)^3(1/2)^3 + (5/6)^4(1/2)^4 + \cdots \\
&= (5/12)\sum_{j=0}^{\infty}(5/12)^j \\
&= \frac{5/12}{1-5/12} \\
&= 5/7.
\end{aligned}$$

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Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 3: Random Variables, Probability and Distributions

STAT/MA 41600
Practice Problems: September 10, 2014
Solutions by Mark Daniel Ward

1. Harmonicas. Since X is a waiting time, then X takes value in the interval $[0, \infty)$, so X is a continuous random variable.

Since Y is a nonnegative integer, i.e., Y takes values 0, 1, 2, 3, ..., then Y is a discrete random variable.

2. Choosing a page at random.
(a.) Find $P(X = 122)$. We have $P(X = 122) = 1/1000$.
(b.) Find $P(X = 977)$. We have $P(X = 977) = 1/1000$.
(c.) Find $P(X = -2)$. We have $P(X = -2) = 0$.
(d.) Find $P(X = 1003)$. We have $P(X = 1003) = 0$.
(e.) When x is an integer between 0 and 999, find $P(X = x)$. We have $P(X = x) = 1/1000$.
(f.) Find $P(X \leq 3)$. We have $P(X \leq 3) = 4/1000$.
(g.) Find $P(X \leq 122)$. We have $P(X \leq 122) = 123/1000$.
(h.) Find $P(12 \leq X \leq 17)$. We have $P(12 \leq X \leq 17) = 6/1000$.
(i.) Find $P(X > 122)$. We have $P(X > 122) = 1 - P(X \leq 122) = 1 - 123/1000 = 877/1000$.
(j.) Find $P(X = 15.73)$. We have $P(X = 15.73) = 0$.
(k.) Find $P(X \leq 15.73)$. We have $P(X \leq 15.73) = 16/1000$.

3. Gloves. a. When j is a positive integer, $P(X = j) = (\frac{4}{5})^{j-1}(\frac{1}{5})$.
b. When j is a positive integer with $1 \leq j \leq 5$, then $P(X = j) = 1/5$.

4. Three dice. Since the sum of the three dice is an integer between 3 and 18, then $P(X = j)$ is strictly positive for integers j with $3 \leq j \leq 18$.

5. Pick two cards. Let A_1, A_2 be (respectively) the events that the first, second card is a face card. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

$$P(X = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (40/52)(39/51) = 30/51.$$

Use the same A_1, A_2 as above. Then

$$P(X = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (12/52)(40/51) + (40/52)(12/51) = 80/221.$$

Use the same A_1, A_2 as above. Then

$$P(X = 2) = P(A_1 \cap A_2) = (12/52)(11/51) = 11/221.$$

STAT/MA 41600
Practice Problems: September 12, 2014
Solutions by Mark Daniel Ward

1. Butterflies. Let A, B, C be, respectively, the events that Alice, Bob, Charlie finds a butterfly. Then

$$\begin{aligned}
p_X(0) &= P(A^c)P(B^c)P(C^c) = 0.3424 \\
p_X(1) &= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C) = 0.4644 \\
p_X(2) &= P(A)P(B)P(C^c) + P(A)P(B^c)P(C) + P(A^c)P(B)P(C) = 0.1741 \\
p_X(3) &= P(A)P(B)P(C) = 0.0191
\end{aligned}$$

2. Appetizers.

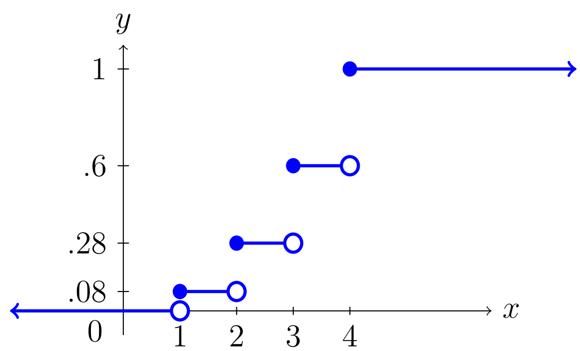

Figure 1: the CDF $F_X(x)$ for the price X of a randomly-chosen appetizer.

3. Wastebasket basketball. The mass of X is

$$
\begin{aligned}
p_X(1) &= 1/3 = .3333\\
p_X(2) &= (2/3)(1/3) = 2/9 = .2222\\
p_X(3) &= (2/3)^2(1/3) = 4/27 = .1481\\
p_X(4) &= (2/3)^3(1/3) = 8/81 = .0988\\
p_X(5) &= (2/3)^4(1/3) = 16/243 = .0658\\
p_X(6) &= 1 - 1/3 - 2/9 - 4/27 - 8/81 - 16/243 = 32/243 = .1317
\end{aligned}
$$

So the mass looks like

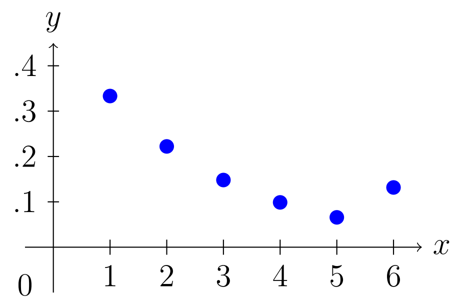

Figure 2: the mass $p_X(x)$ for the number X of attempts required for a basket.

Draw the CDF of X.

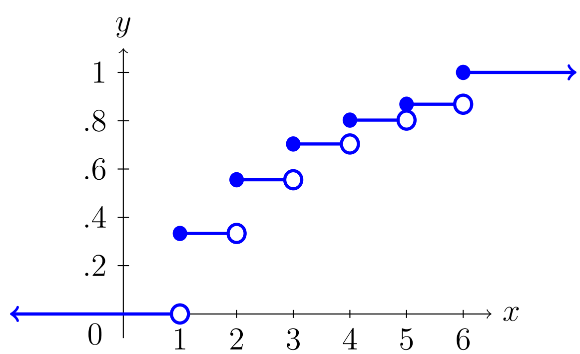

Figure 3: the CDF $F_X(x)$ for the number X of attempts required for a basket.

4. Two 4-sided dice. The mass of X is

$$\begin{aligned} p_X(2) &= 1/16 = .0625 \\ p_X(3) &= 2/16 = .125 \\ p_X(4) &= 3/16 = .1875 \\ p_X(5) &= 4/16 = .25 \\ p_X(6) &= 3/16 = .1875 \\ p_X(7) &= 2/16 = .125 \\ p_X(8) &= 1/16 = .0625 \end{aligned}$$

So the mass looks like

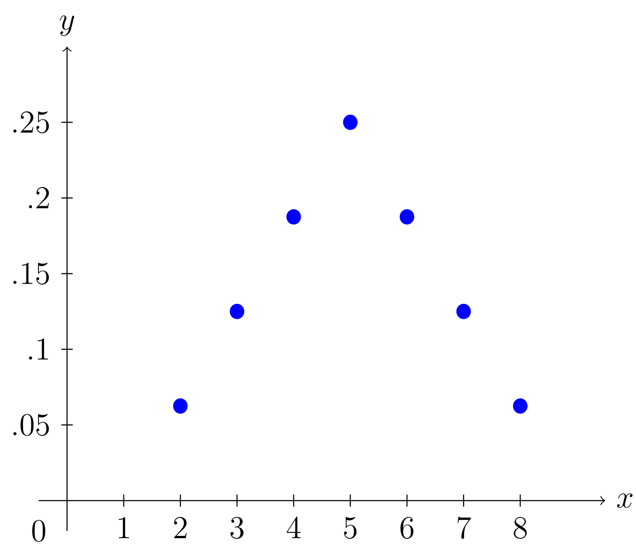

Figure 4: the mass $p_X(x)$ for the sum X of two 4-sided dice.

Draw the CDF of X.

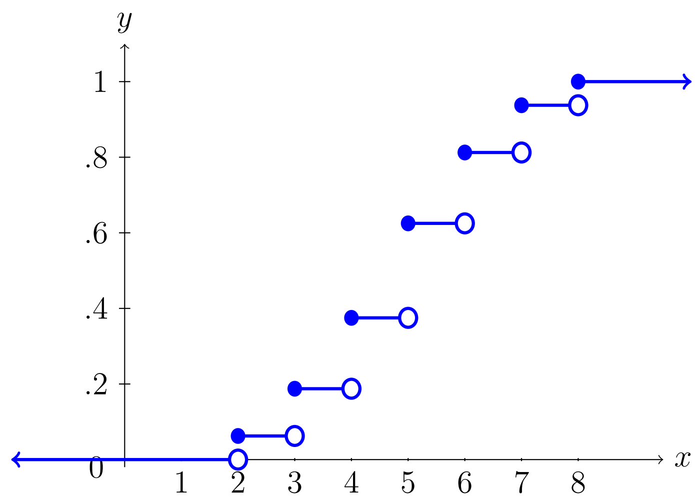

Figure 5: the CDF $F_X(x)$ for the sum X of two 4-sided dice.

5. Pick two cards. As discussed in Problem Set 7, the mass of X is:

$$p_X(0) = 30/51 = .5882, \qquad p_X(1) = 80/221 = .3620, \qquad p_X(2) = 11/221 = .0498.$$

So the CDF of X is

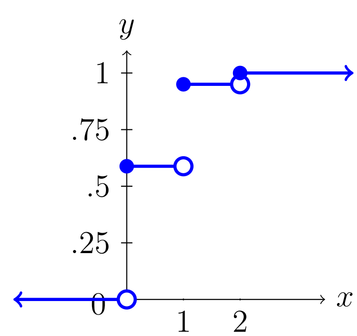

Figure 6: the CDF $F_X(x)$ for the number X of face cards.

STAT/MA 41600
Practice Problems: September 15, 2014
Solutions by Mark Daniel Ward

1. Butterflies. Using the work from Problem Set 8, we have

$$p_{X,Y}(0,3) = .3424, \qquad p_{X,Y}(1,2) = .4644, \qquad p_{X,Y}(2,1) = .1741, \qquad p_{X,Y}(3,0) = .0191,$$

and otherwise $p_{X,Y}(x,y) = 0$.

2. Dependence/independence among dice rolls. Yes, X and Y are independent because for positive integers x and y, we have

$$p_{X,Y}(x,y) = \big((5/6)^{x-1}(1/6)\big)\big((5/6)^{y-1}(1/6)\big) = p_X(x)p_Y(y)$$

and otherwise $p_{X,Y}(x,y) = 0 = p_X(x)p_Y(y)$. So X and Y are independent.

3. Wastebasket basketball. Using the results from Problem Set 8, we have $p_Y(1) = 1/3 + 2/9 + 4/27 = 19/27$, and thus $p_Y(0) = 8/27$. So we have

$$p_{X|Y}(1 \mid 0) = \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(2 \mid 0) = \frac{p_{X,Y}(2,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(3 \mid 0) = \frac{p_{X,Y}(3,0)}{p_Y(0)} = \frac{0}{8/27} = 0$$
$$p_{X|Y}(4 \mid 0) = \frac{p_{X,Y}(4,0)}{p_Y(0)} = \frac{8/81}{8/27} = 1/3$$
$$p_{X|Y}(5 \mid 0) = \frac{p_{X,Y}(5,0)}{p_Y(0)} = \frac{16/243}{8/27} = 2/9$$
$$p_{X|Y}(6 \mid 0) = \frac{p_{X,Y}(6,0)}{p_Y(0)} = \frac{32/243}{8/27} = 4/9$$

and

$$p_{X|Y}(1 \mid 1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/3}{19/27} = 9/19$$
$$p_{X|Y}(2 \mid 1) = \frac{p_{X,Y}(2,1)}{p_Y(1)} = \frac{2/9}{19/27} = 6/19$$
$$p_{X|Y}(3 \mid 1) = \frac{p_{X,Y}(3,1)}{p_Y(1)} = \frac{4/27}{19/27} = 4/19$$
$$p_{X|Y}(4 \mid 1) = \frac{p_{X,Y}(4,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$
$$p_{X|Y}(5 \mid 1) = \frac{p_{X,Y}(5,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$
$$p_{X|Y}(6 \mid 1) = \frac{p_{X,Y}(6,1)}{p_Y(1)} = \frac{0}{19/27} = 0$$

4. Two 4-sided dice. In general, for $1 \le x < y \le 4$,

$$\begin{aligned} p_{X,Y}(x,y) &= P(\{\max = y \text{ and } \min = x\}) \\ &= P(\{\text{die values } x, y\}) + P(\{\text{die values } y, x\}) \\ &= 2/16, \end{aligned}$$

and for $1 \le x = y \le 4$,

$$\begin{aligned} p_{X,Y}(x,y) &= P(\{\max = y = x = \min\}) \\ &= P(\{\text{die values } x, x\}) \qquad \text{since } x \text{ and } y \text{ are the same in this case} \\ &= 1/16, \end{aligned}$$

and

$$p_{X,Y}(x,y) = 0 \qquad \text{otherwise.}$$

We have

$F_{X,Y}(x,y)$	$y = 1$	$y = 2$	$y = 3$	$y = 4$
$x = 1$	1/16	3/16	5/16	7/16
$x = 2$	1/16	4/16	8/16	12/16
$x = 3$	1/16	4/16	9/16	15/16
$x = 4$	1/16	4/16	9/16	16/16

5. Pick two cards. The random variables X and Y are dependent. As an example, we know $P(X = 2) = 11/221$; this was established back in Problem Set 7. On the other hand, given $Y = 2$, then both of the cards that you selected are 10's, so they cannot be face cards, and thus

$$P(X = 2 \mid Y = 2) = 0.$$

So we have

$$P(X = 2 \mid Y = 2) \ne P(X = 2).$$

So X and Y are dependent random variables.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 4: Expected Values

STAT/MA 41600
Practice Problems: September 17, 2014
Solutions by Mark Daniel Ward

1. Butterflies. The mass of X is

$$p_X(0) = 0.3424, \qquad p_X(1) = 0.4644, \qquad p_X(2) = 0.1741, \qquad p_X(3) = 0.0191,$$

so the expected value of X is

$$\mathbb{E}(X) = (0)(0.3424) + (1)(0.4644) + (2)(0.1741) + (3)(0.0191) = .87$$

2. Dependence/independence among dice rolls. We have $P(X = j) = (5/6)^{j-1}(1/6)$ for each positive integer j. So

$$\begin{aligned}\mathbb{E}(X) &= (1)P(X = 1) + (2)P(X = 2) + (3)P(X = 3) + \cdots \\ &= \sum_{j=1}^{\infty} j(5/6)^{j-1}(1/6) \\ &= \frac{1}{6}\sum_{j=1}^{\infty} j(5/6)^{j-1}\end{aligned}$$

We notice that $j(5/6)^{j-1}$ is the derivative of x^j (with respect to x), evaluated at $x = 5/6$. So we rewrite the equation above as follows:

$$\mathbb{E}(X) = \frac{1}{6}\sum_{j=1}^{\infty} \frac{d}{dx}x^j\bigg|_{x=5/6} = \frac{1}{6}\frac{d}{dx}\sum_{j=1}^{\infty} x^j\big|_{x=5/6}$$

We use our scrap paper to compute:

$$\sum_{j=1}^{\infty} x^j = x\sum_{j=0}^{\infty} x^j = \frac{x}{1-x}$$

So we get

$$\begin{aligned}\mathbb{E}(X) &= \frac{1}{6}\,\frac{d}{dx}\frac{x}{1-x}\bigg|_{x=5/6} \\ &= \frac{1}{6}\,\frac{(1-x)(1) - (x)(-1)}{(1-x)^2}\bigg|_{x=5/6} \\ &= \frac{1}{6}\,\frac{1}{(1-x)^2}\bigg|_{x=5/6} \\ &= \frac{1}{6}\frac{1}{\left(1-\frac{5}{6}\right)^2} \\ &= \frac{1}{6}\frac{1}{(1/6)^2} \\ &= 6\end{aligned}$$

So the student expects to roll 6 times to get the first 4.

The same reasoning shows that $\mathbb{E}(Y) = 6$ too.

3. Wastebasket basketball. As in Problem Set 8, The mass of X is

$p_X(1) = 1/3, \quad p_X(2) = 2/9, \quad p_X(3) = 4/27, \quad p_X(4) = 8/81, \quad p_X(5) = 16/243, \quad p_X(6) = 32/243.$

So the expected value of X is

$\mathbb{E}(X) = (1)(1/3)+(2)(2/9)+(3)(4/27)+(4)(8/81)+(5)(16/243)+(6)(32/243) = 665/243 = 2.7366$

4. Two 4-sided dice. We compute

$$\begin{aligned}\mathbb{E}(X) &= (1)p_{X,Y}(1,1) + (1)p_{X,Y}(1,2) + (1)p_{X,Y}(1,3) + (1)p_{X,Y}(1,4) \\ &\qquad + (2)p_{X,Y}(2,2) + (2)p_{X,Y}(2,3) + (2)p_{X,Y}(2,4) \\ &\qquad\qquad + (3)p_{X,Y}(3,3) + (3)p_{X,Y}(3,4) \\ &\qquad\qquad\qquad + (4)p_{X,Y}(4,4) \\ &= (1)(1/16) + (1)(2/16) + (1)(2/16) + (1)(2/16) \\ &\qquad + (2)(1/16) + (2)(2/16) + (2)(2/16) \\ &\qquad\qquad + (3)(1/16) + (3)(2/16) \\ &\qquad\qquad\qquad + (4)(1/16) \\ &= 30/16 \\ &= 15/8\end{aligned}$$

We compute

$$\begin{aligned}\mathbb{E}(Y) &= (1)p_{X,Y}(1,1) + (2)p_{X,Y}(1,2) + (3)p_{X,Y}(1,3) + (4)p_{X,Y}(1,4) \\ &\qquad + (2)p_{X,Y}(2,2) + (3)p_{X,Y}(2,3) + (4)p_{X,Y}(2,4) \\ &\qquad\qquad + (3)p_{X,Y}(3,3) + (4)p_{X,Y}(3,4) \\ &\qquad\qquad\qquad + (4)p_{X,Y}(4,4) \\ &= (1)(1/16) + (2)(2/16) + (3)(2/16) + (4)(2/16) \\ &\qquad + (2)(1/16) + (3)(2/16) + (4)(2/16) \\ &\qquad\qquad + (3)(1/16) + (4)(2/16) \\ &\qquad\qquad\qquad + (4)(1/16) \\ &= 50/16 \\ &= 25/8\end{aligned}$$

5. Pick two cards. From Problem Set 7, we know that the mass of X is

$$P(X = 0) = 30/51, \qquad P(X = 1) = 80/221, \qquad P(X = 2) = 11/221.$$

So

$$\mathbb{E}(X) = (0)(30/51) + (1)(80/221) + (2)(11/221) = 6/13.$$

Let A_1, A_2 be (respectively) the events that the first, second card is a 10 card. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

$$P(Y = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (48/52)(47/51) = 188/221.$$

Using the same A_1, A_2 as above,

$$P(Y = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (4/52)(48/51) + (48/52)(4/51) = 32/221.$$

Using the same A_1, A_2 as above,

$$P(Y = 2) = P(A_1 \cap A_2) = (4/52)(3/51) = 1/221.$$

So

$$\mathbb{E}(Y) = (0)(188/221) + (1)(32/221) + (2)(1/221) = 2/13.$$

STAT/MA 41600
Practice Problems: September 19, 2014
Solutions by Mark Daniel Ward

1. Butterflies. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = .17 + .25 + .45 = .87$.

2. Dependence/independence among dice rolls. We have

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{j=1}^{\infty} X_j\right) = \sum_{j=1}^{\infty} \mathbb{E}(X_j) = \sum_{j=1}^{\infty} (5/6)^{j-1} = \frac{1}{1 - \frac{5}{6}} = 6$$

The same steps show $\mathbb{E}(Y) = \mathbb{E}\left(\sum_{j=1}^{\infty} Y_j\right) = \sum_{j=1}^{\infty} \mathbb{E}(Y_j) = \sum_{j=1}^{\infty} (5/6)^{j-1} = \frac{1}{1-\frac{5}{6}} = 6$.

3. Wastebasket basketball. We have

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) + \mathbb{E}(X_5) + \mathbb{E}(X_6) \\
&= 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 = \frac{665}{243} = 2.7366
\end{aligned}$$

4. Two 4-sided dice. We have

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) = 1 + \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = 15/8$$

5. Pick two cards. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 12/52 + 12/52 = 6/13$.

Before looking at the cards, put one in your left hand and one in your right hand. Let Y_1 and Y_2 indicate, respectively, whether the cards in your left and right hands (respectively) are 10's. Then $Y = Y_1 + Y_2$. Find the expected value of Y by finding the expected value of the sum of the indicator random variables.

We have $\mathbb{E}(Y) = \mathbb{E}(Y_1 + Y_2) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) = 4/52 + 4/52 = 2/13$.

STAT/MA 41600
Practice Problems: September 22, 2014
Solutions by Mark Daniel Ward

1a. Variance of an Indicator. We have $\mathbb{E}(X^2) = 1^2(p) + 0^2(1-p) = p$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = p - p^2 = p(1-p)$.

1b. Butterflies. *Method #1:* Write X as the sum of three indicator random variables, X_1, X_2, X_3 that indicate whether Alice, Bob, Charlotte (respectively) found a butterfly. Then $X = X_1 + X_2 + X_3$. Since X_1, X_2, X_3 are independent, then $\text{Var}(X) = \text{Var}(X_1 + X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = (.17)(.83) + (.25)(.75) + (.45)(.55) = .5761$.

Method #2: The mass and expected value of X were given in Problem Set 10:

The mass of X is

$$p_X(0) = 0.3424, \qquad p_X(1) = 0.4644, \qquad p_X(2) = 0.1741, \qquad p_X(3) = 0.0191,$$

so the expected value of X is

$$\mathbb{E}(X) = (0)(0.3424) + (1)(0.4644) + (2)(0.1741) + (3)(0.0191) = .87.$$

The expected value of X^2 is

$$\mathbb{E}(X^2) = (0^2)(0.3424) + (1^2)(0.4644) + (2^2)(0.1741) + (3^2)(0.0191) = 1.33.$$

So the variance of X is $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1.3327 - (.87)^2 = .58$.

2. Appetizers. *Method #1:* The expected value of X is:

$$\mathbb{E}(X) = (1)(.08) + (2)(.20) + (3)(.32) + (4)(.40) = 3.04,$$

and

$$\mathbb{E}(X^2) = 1^2(.08) + 2^2(.20) + 3^2(.32) + 4^2(.40) = 10.16,$$

so

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10.16 - (3.04)^2 = .9184.$$

So

$$\text{Var}(Y) = \text{Var}(1.07X) = (1.07)^2\,\text{Var}(X) = (1.07)^2(.9184) = 1.0515.$$

Method #2: The expected value of Y is:

$$\mathbb{E}(Y) = (1.07)(1)(.08) + (1.07)(2)(.20) + (1.07)(3)(.32) + (1.07)(4)(.40) = 3.2528,$$

and

$$\mathbb{E}(Y^2) = ((1.07)(1))^2(.08) + ((1.07)(2))^2(.20) + ((1.07)(3))^2(.32) + ((1.07)(4))^2(.40) = 11.632184,$$

so

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 11.632184 - (3.2528)^2 = 1.0515.$$

3. Gloves. As discussed in Problem Set 3, X has mass $1/5$ on the values $1, 2, 3, 4, 5$. So

$$\mathbb{E}(X) = (1)(1/5) + (2)(1/5) + (3)(1/5) + (4)(1/5) + (5)(1/5) = 3.$$

Find $\mathbb{E}(X^2)$.
Again using the mass of X, we have

$$\mathbb{E}(X^2) = 1^2(1/5) + 2^2(1/5) + 3^2(1/5) + 4^2(1/5) + 5^2(1/5) = 11.$$

We have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 11 - 3^2 = 2.$$

Again using the mass of X, we have

$$\mathbb{E}(X^3) = 1^3(1/5) + 2^3(1/5) + 3^3(1/5) + 4^3(1/5) + 5^3(1/5) = 45.$$

4. Two 4-sided dice. [Caution: If X_j is an indicator of whether the minimum of the two dice is "j or greater"—as in the previous homework—then $X = X_1 + X_2 + X_3 + X_4$, but the X_j's are dependent. So we cannot just sum the variances. We need to find the mass of X and then compute the expected value and variance by hand.]

The mass of X is

$$p_X(1) = 7/16, \qquad p_X(2) = 5/16, \qquad p_X(3) = 3/16, \qquad p_X(4) = 1/16.$$

So

$$\mathbb{E}(X) = 1(7/16) + 2(5/16) + 3(3/16) + 4(1/16) = 15/8,$$

and

$$\mathbb{E}(X^2) = 1^2(7/16) + 2^2(5/16) + 3^2(3/16) + 4^2(1/16) = 35/8,$$

and thus

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 35/8 - (15/8)^2 = 55/64 = .859375.$$

5. Pick two cards. As in Problem Set 7, the mass of X is

$$p_X(0) = 30/51, \qquad p_X(1) = 80/221, \qquad p_X(2) = 11/221.$$

So

$$\mathbb{E}(X) = (0)(30/51) + (1)(80/221) + (2)(11/221) = 6/13,$$

and

$$\mathbb{E}(X^2) = (0^2)(30/51) + (1^2)(80/221) + (2^2)(11/221) = 124/221,$$

and thus

$$\text{Var}(X) = 124/221 - (6/13)^2 = 1000/2873 = .3481.$$

First we find the mass of Y. Let A_1, A_2 be (respectively) the events that the first, second card is a 10. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

$$P(Y = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (48/52)(47/51) = 188/221,$$

and

$$P(Y = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (4/52)(48/51) + (48/52)(4/51) = 32/221,$$

and

$$P(Y = 2) = P(A_1 \cap A_2) = (4/52)(3/51) = 1/221.$$

So

$$\mathbb{E}(Y) = (0)(188/221) + (1)(32/221) + (2)(1/221) = 2/13,$$

and

$$\mathbb{E}(Y^2) = (0^2)(188/221) + (1^2)(32/221) + (2^2)(1/221) = 36/221,$$

and thus

$$\text{Var}(Y) = 36/221 - (2/13)^2 = 400/2873 = .1392.$$

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 5: Models of Discrete Random Variables I

STAT/MA 41600
Practice Problems: September 24, 2014
Solutions by Mark Daniel Ward

1. Winnings and Losing. (a.) His expected gain/loss is $(.4)(5) + (.6)(-4) = -0.40$.
(b.) The variance of his gain or loss is $(.4)(5)^2 + (.6)(-4)^2 - (-0.40)^2 = 19.44$.

(c.) We have $a = 9$ and $b = -4$, so $Y = 9X - 4$. Thus $\mathbb{E}(Y) = 9\mathbb{E}(X) - 4 = 9(.4) - 4 = -0.40$ and $\mathrm{Var}(Y) = 9^2\,\mathrm{Var}(X) = (9^2)(.6)(.4) = 19.44$.

2. Telemarketers. (a.) The probability that the next caller is a telemarketer is $1/8$.

(b.) The probability that the 3rd caller is a telemarketer is $1/8$.

(c.) Let X indicate whether the next caller is a telemarketer. Then we lose $30X$ seconds on the next phone call, so we expected to lose $\mathbb{E}(30X) = 30\mathbb{E}(X) = (30)(1/8) = 30/8 = 15/4 = 3.75$ seconds on the next phone call.

(d.) Again, let X indicate whether the next caller is a telemarketer. Then we lose $30X$ seconds on the next phone call, so the variance of the time lost is $\mathrm{Var}(30X) = 900\,\mathrm{Var}(X) = (900)(1/8)(7/8) = 1575/16 = 98.4375$, and the standard deviation of the time lost is $\sigma_{30X} = \sqrt{98.4375} = 9.9216$.

3. Dating. We have $\mathbb{E}(X_j) = P(X_j = 1)$, which is the probability that the first $j-1$ attempts are unsuccessful, i.e., $(.93)^{j-1}$. So $\mathbb{E}(X) = \sum_{j=1}^{\infty}(.93)^{j-1} = \frac{1}{1-.93} = 1/.07 = 100/7 = 14.29$.

4. Studying. We have $\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_{30}) = .65 + .65 + \cdots + .65 = (30)(.65) = 19.50$.

Since the X_j's are independent, then $\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_{30}) = (.65)(.35) + (.65)(.35) + \cdots + (.65)(.35) = (30)(.65)(.35) = 6.825$.

5. Shoes. (a.) Altogether $(.20)(15) + (.10)(40) = 7$ out of the $15 + 40 = 55$ shoes are sandals, so the probability is $7/55 = .1272$.

(b.) Exactly 15 of the 55 shoes belong to Anne, so the probability is $15/55 = 3/11 = .2727$.

(c.) There are exactly 7 sandals; each is equally-likely to be chosen, and 3 of them are Anne's, so the probability is $3/7 = .4286$.

STAT/MA 41600
Practice Problems: September 24, 2014
Solutions by Mark Daniel Ward

1. Winnings and Losing. (a.) His expected total gain/loss is $\mathbb{E}(9X - 40) = 9\mathbb{E}(X) - 40 = 9(4) - 40 = -4$.

(b.) The variance of his total gain or loss is $\text{Var}(9X-40) = 81\,\text{Var}(X) = (81)(10)(.6)(.4) = 194.4$.

(c.) His winnings are $9X - 40$, so the probability that he wins \$32 or more is

$$\begin{aligned} P(9X - 40 \geq 32) &= P(9X \geq 72) = P(X \geq 8) \\ &= \binom{10}{8}(.4)^8(.6)^2 + \binom{10}{9}(.4)^9(.6)^1 + \binom{10}{10}(.4)^{10}(.6)^0 \\ &= 45\left(\frac{2304}{9765625}\right) + 10\left(\frac{1536}{9765625}\right) + 1\left(\frac{1024}{9765625}\right) \\ &= \frac{120064}{9765625} \\ &= 0.012295. \end{aligned}$$

2. Telemarketers. (a.) Since X is Binomial with $n = 3$ and $p = 1/8$, then

$$p_X(x) = \binom{3}{x}(1/8)^x(7/8)^{3-x} \qquad \text{when } x \text{ is 0, 1, 2, or 3,}$$

and $p_X(x) = 0$ otherwise. So

$$p_X(0) = \frac{343}{512}, \qquad p_X(1) = \frac{147}{512}, \qquad p_X(2) = \frac{21}{512}, \qquad p_X(3) = \frac{1}{512}.$$

3. Dating. (a.) Let X be the number of people who accept the invitation. So X is Binomial with $n = 20$ and $p = .07$. So the probability of $X \geq 3$ is

$$\begin{aligned} P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - \binom{20}{0}(.07)^0(.93)^{20} - \binom{20}{1}(.07)^1(.93)^{19} - \binom{20}{2}(.07)^2(.93)^{18} \\ &= 1 - (1)(0.2342) - (20)(0.01763) - (190)(0.001327) \\ &= 0.161 \end{aligned}$$

(b.) Since X is Binomial($n = 20,\ p = .07$), then $\mathbb{E}(X) = np = (20)(.07) = 1.40$.

(c.) Since X is Binomial($n = 20,\ p = .07$), then $\text{Var}(X) = npq = (20)(.07)(.93) = 1.3020$.

4. Dining Hall. (a.) Since X, Y, Z are independent Binomials with the same p, then $X + Y + Z$ is Binomial too, with $n = 7 + 7 + 7 = 21$ and with the same $p = .65$.

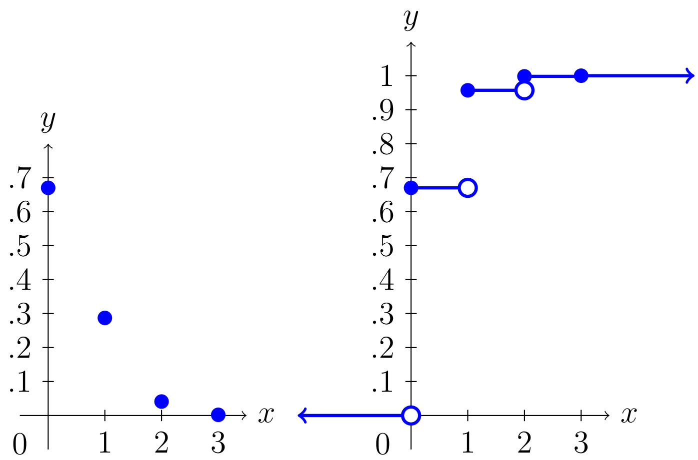

Figure 1: For X telemarketers, the mass $p_X(x) = P(X = x)$ and CDF $F_X(x) = P(X \leq x)$

Since $X + Y + Z$ is Binomial with $n = 21$ and $p = .65$, then $\mathrm{Var}(X + Y + Z) = npq = (21)(.65)(.35) = 4.7775$.

5. Hearts. (a.) Let X_j denote the jth card that is drawn. Then $\mathbb{E}(X_j) = 13/52 = 1/4$. So $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_7) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_j) = 1/4 + \cdots + 1/4 = 7/4 = 1.75$.

(b.) No, X is not binomial because the X_j's are dependent.

STAT/MA 41600
Practice Problems: September 26, 2014
Solutions by Mark Daniel Ward

1. Winnings and Losing. (a.) Since X is Geometric with probability of success 0.40, he expects to play $\mathbb{E}(X) = 1/0.40 = 2.5$ games.

(b.) Since X is Geometric with probability of success 0.40, the variance of the number of games he plays is $\text{Var}(X) = 0.60/(0.40^2) = 3.75$.

(c.) The probability that he plays 4 or more games is equal to the probability that the first three games are all losses, i.e., $(.6)^3 = 0.216$.

2. Winnings and Losing (continued). (a.) His gain or loss is $Y = 5 + (-4)(X - 1) = 9 - 4X$, since he wins 1 game and loses $X - 1$ games.]

(b.) His expected gain/loss is $\mathbb{E}(Y) = \mathbb{E}(9 - 4X) = 9 - 4\mathbb{E}(X) = 9 - 4(2.5) = -1$.

(c.) The variance of his gain/loss is $\text{Var}(Y) = \text{Var}(9 - 4X) = 16\,\text{Var}(X) = 16(3.75) = 60$.

3. Telemarketers. Here X is Geometric with probability of success 1/8, because "success" denotes a call from the telemarketer. So $X > n$ if the first n calls are unsuccessful, i.e., if the first n calls are not telemarketers. So $P(X > n) = (7/8)^n$.

4. Dating. (a.) Since X is Geometric with probability of success .07, then $\mathbb{E}(X) = 1/.07 = 100/7 = 14.29$.

(b.) Since X is Geometric with probability of success .07, then $\text{Var}(X) = .93/(.07)^2 = 189.8$.

(c.) Since X is memoryless, then given $X > 3$, the remaining $Y = X - 3$ people we need to call is also Geometric with probability of success .07. So the mass of Y given $X > 3$ is $P(Y = 3 \mid X > 3) = (.93)^{y-1}(.07)$ for integers $y \geq 1$, and $p_Y(y) = 0$ otherwise.

5. Hearts. (a.) Since X is Geometric with probability of success 1/4, then you expect to draw $\mathbb{E}(X) = 1/(1/4) = 4$ cards to see the first heart.

(b.) Since X is memoryless, then since we are given that the first 5 cards are not hearts, it follows that the additional number of cards (after the first five are drawn) is also Geometric, with probability of success 1/4. So we expect to draw an additional $1/(1/4) = 4$ cards to see the first heart (after those first five are already drawn). (I.e., we expect that the first heart appears after 9 cards altogether.)

STAT/MA 41600
Practice Problems: September 29, 2014
Solutions by Mark Daniel Ward

1. Quidditch Training.
(a.) Since X is Negative Binomial with $r = 4$ successes and with $p = 0.15$ on each trial, then the mass of X is $p_X(x) = \binom{x-1}{3}(.85)^{x-4}(.15)^4$, for $x = 4, 5, 6, \ldots$, and $p_X(x) = 0$ otherwise.

(b.) The probability is $p_X(12) = \binom{11}{3}(.85)^8(.15)^4, = 0.02276$.

(c.) The expected value is $\mathbb{E}(X) = r/p = 4/0.15 = 26.67$.

2. Horcruxes.
(a.) Since X is Negative Binomial with $r = 7$ and $p = 1/3$, then $\mathbb{E}(X) = r/p = 7/(1/3) = 21$.

(b.) As before, using Negative Binomial parameters, $\operatorname{Var} X = qr/p^2 = (2/3)7/(1/3)^2 = 42$.

(c.) The mass of X is $p_X(x) = \binom{x-1}{6}(2/3)^{x-7}(1/3)^7$, for $x = 7, 8, 9, \ldots$, and $p_X(x) = 0$ otherwise. So $p_X(9) = \binom{8}{6}(2/3)^2(1/3)^7 = 112/19683 = 0.00569$.

3. Mandrakes.

(a.) The expected number is $3/0.02 = 150$.

(b.) The variance is $(.98)(3)/(0.02)^2 = 7350$.

4. Divination.

(a.) Lavender earns $Y = (100)(5) - (15)(X - 5) = 575 - 15X$.

(b.) Her expected earnings are $\mathbb{E}(Y) = \mathbb{E}(575 - 15X) = 575 - 15\mathbb{E}(X) = 575 - 15(5/0.12) = -50$.

(c.) Her earnings have variance $\operatorname{Var}(Y) = \operatorname{Var}(575 - 15X) = 15^2 \operatorname{Var}(X) = 15^2 \frac{(0.88)(5)}{(.12)^2} = 68750$.

5. Spells.

(a.) The expected value of X is $\mathbb{E}(X) = 5/.30 + 3/.30 + 20/.30 = 93.33$. This can also be checked by noticing that X is a Negative Binomial random variable with $r = 28$ and $p = .30$, so $\mathbb{E}(X) = r/p = 28/.3 = 93.33$.

(b.) We have $\operatorname{Var}(X) = (.7)(5)/(.30)^2 + (.7)(3)/(.30)^2 + (.7)(20)/(.30)^2 = 217.78$. Also: X is a Negative Binomial with $r = 28$ and $p = .30$, so $\operatorname{Var}(X) = qr/p^2 = (.7)(28)/(.3)^2 = 217.78$.

PurdueX: 416.1x
Probability: Basic Concepts & Discrete Random Variables

Solution to the problem sets

Unit 6: Models of Discrete Random Variables II

STAT/MA 41600
Practice Problems: October 1, 2014
Solutions by Mark Daniel Ward

1. Hungry customers.

a. The number X of arrivals in the next 10 minutes is Poisson with average of 5 people. So the probability of exactly 3 arrivals is $P(X = 3) = e^{-5}5^3/3! = .1404$.

b. Again, the number X of arrivals in the next 10 minutes is Poisson with average of 5 people. So the probability of no arrivals is $P(X = 0) = e^{-5}5^0/0! = .0067$.

c. Again, the number X of arrivals in the next 10 minutes is Poisson with average of 5 people. So the probability of at least 3 arrivals is $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - e^{-5}5^0/0! - e^{-5}5^1/1! - e^{-5}5^2/2! = .8753$.

2. Errors in Dr. Ward's book.

a. The number X of errors in the next 100 pages is Poisson with average $(100)(.04) = 4$.

b. Since the number X of errors in the next 100 pages is Poisson with average $(100)(.04) = 4$, then the probability of exactly 5 errors is $P(X = 5) = e^{-4}4^5/5! = .1563$.

3. Telemarketers.

a. The number X of telemarketers per day is Poisson with average $\lambda = 3/7 = .4285$ per day. So the mass is $P(X = j) = e^{-3/7}(3/7)^j/j!$ for $j = 0, 1, 2, 3, \ldots$, and $P(X = j) = 0$ otherwise.

b. Since the number X of telemarketers per day is Poisson with average $\lambda = 3/7 = .4285$ per day, then the probability that none of them call your house on a given day is $P(X = 0) = e^{-\lambda}\lambda^0/0! = e^{-\lambda} = .6514$.

c. Since the number X of telemarketers per day is Poisson with average $\lambda = 3/7 = .4285$ per day, then the probability that exactly 2 of them call your house on a given day is $P(X = 2) = e^{-\lambda}\lambda^2/2! = .0598$.

4. Superfans.

a. Altogether we expert 14 such fans per hour, so we expect $(3)(14) = 42$ of them during the next 3 hours.

b. Since the number X of such fans has average 14 per hour, then the average during the next 20 minutes is $(14)(20/60) = 4.6667$. So the probability of exactly one such fan is $P(X = 1) = e^{-4.6667}(4.6667)^1/1! = 0.04388$.

5. Shoppers.

a. The number X of men in a 10-second period is Poisson with average $(12)(10/60) = 2$, so the probability of 1 man in the next ten seconds is $P(X = 1) = e^{-2}2^1/1! = 2e^{-2}$. The number Y of women in a 10-second period is Poisson with average $(15)(10/60) = 5/2$, so the probability of 2 women in the next ten seconds is $P(Y = 2) = e^{-5/2}(5/2)^2/2! = \frac{25}{8}e^{-5/2}$. So the desired probability, since X and Y are independent, is $(2e^{-2})(\frac{25}{8}e^{-5/2}) = \frac{25}{4}e^{-9/2} = 0.0694$.

b. The number of people per minute is Poisson with average 27 per minute. So the number Z of people in a 5-minute period is Poisson with mean $\mathbb{E}(Z) = (5)(27) = 135$. The expected value and variance of a Poisson is always the same, so $\mathrm{Var}(Z) = 135$ too.

STAT/MA 41600
Practice Problems: October 3, 2014
Solutions by Mark Daniel Ward

1. Hungry customers.

a. The number X of people in the survey who are eating pizza is Hypergeometric with $M = 7$, $N = 12$, and $n = 3$. So the mass of X is $p_X(x) = \frac{\binom{7}{x}\binom{5}{3-x}}{\binom{12}{3}}$.

b. The values are

$$p_X(0) = \frac{\binom{7}{0}\binom{5}{3-0}}{\binom{12}{3}} = 1/22,$$
$$p_X(1) = \frac{\binom{7}{1}\binom{5}{3-1}}{\binom{12}{3}} = 7/22,$$
$$p_X(2) = \frac{\binom{7}{2}\binom{5}{3-2}}{\binom{12}{3}} = 21/44,$$
$$p_X(3) = \frac{\binom{7}{3}\binom{5}{3-3}}{\binom{12}{3}} = 7/44.$$

c. We can write the average as $(0)(1/22) + (1)(7/22) + (2)(21/44) + (3)(7/44) = 7/4$, but it is perhaps easier to note that each person in the survey eats pizza with probability $7/12$, so the average number of survey participants eating pizza is $7/12 + 7/12 + 7/12 = (3)(7/12) = 21/12 = 7/4$.

2. Harmonicas.

a. The number X of Deluxe harmonicas is Hypergeometric with $M = 7$, $N = 19$, and $n = 8$. So $\mathbb{E}(X) = (8)(7/19) = 56/19 = 2.9474$. This can also be seen by the fact that each harmonica selected has a probability $7/19$ of being a Deluxe harmonica.

b. Since the number X of Deluxe harmonicas is Hypergeometric with $M = 7$, $N = 19$, and $n = 8$, then $\text{Var}(X) = n\frac{M}{N}\left(1 - \frac{M}{N}\right)\left(\frac{N-n}{N-1}\right) = (8)(7/19)(1-7/19)(11/18) = 1232/1083 = 1.1376$.

c. Since the number X of Deluxe harmonicas is Hypergeometric with $M = 7$, $N = 19$, and $n = 8$, then $P(X = 5) = \frac{\binom{7}{5}\binom{12}{3}}{\binom{19}{8}} = 770/12597 = 0.0611$.

3. Granola bars.

a. The number X of chocolates I grab is Hypergeometric with $M = 6 + 8 = 14$, $N = 24$, and $n = 3$. So $P(X = 2) = \frac{\binom{14}{2}\binom{10}{1}}{\binom{24}{3}} = 455/1012 = 0.4496$.

b. The probability that strictly fewer than 2 are chocolate is $P(X=0)+P(X=1) = \frac{\binom{14}{0}\binom{10}{3}}{\binom{24}{3}} + \frac{\binom{14}{1}\binom{10}{2}}{\binom{24}{3}} = 15/253 + 315/1012 = 375/1012 = 0.3706$.

c. The average number of chocolate granola bars is $n(M/N) = 3(14/24) = 7/4$. This can also be determined by the fact that each bar is chocolate with probability 14/24, so the average number of chocolates is $3(14/24) = 7/4$.

4. Superfans.

a. The exactly probability is $\binom{60{,}000}{8}\left(\frac{1}{10{,}000}\right)^8\left(\frac{9999}{10{,}000}\right)^{59992}$.

b. If $n = 60{,}000$ and $p = 1/10{,}000$, then the probability in part (a) is the probability that a Binomial n, p random variable is equal to 8. Now let X be Poisson with average $\lambda = np = 6$. Then $P(X=8) = \frac{e^{-6}6^8}{8!}$.

c. We have $P(X=8) = \frac{e^{-6}6^8}{8!} = 0.1033$.

5. Shoppers.

a. The exact probability is

$$= \binom{100{,}000}{0}\left(\frac{49{,}999}{50{,}000}\right)^{100{,}000} + \binom{100{,}000}{1}\left(\frac{1}{50{,}000}\right)^1\left(\frac{49{,}999}{50{,}000}\right)^{99{,}999} + \binom{100{,}000}{2}\left(\frac{1}{50{,}000}\right)^2\left(\frac{49{,}999}{50{,}000}\right)^{99{,}998} + \binom{100{,}000}{3}\left(\frac{1}{50{,}000}\right)^3\left(\frac{49{,}999}{50{,}000}\right)^{99{,}997}$$

b. If $n = 100{,}000$ and $p = 1/50{,}000$, then the probability in part (a) is the probability that a Binomial n, p random variable is 3 or less. Now let X be Poisson with average $\lambda = np = 2$. Then $P(X \le 3) = \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!}$.

c. We have $P(X \le 3) = \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!} = 0.8571$.

STAT/MA 41600
Practice Problems: October 6, 2014
Solutions by Mark Daniel Ward

1. Hearts.

a. *Method #1:* We win if 2 or 3 of the 3 cards are hearts. So the probability of winning is $\frac{\binom{39}{1}\binom{13}{2}}{\binom{52}{3}} + \frac{\binom{13}{3}}{\binom{52}{3}} = 64/425 = 0.1506$.
Method #2: The probability of winning is 1 minus the probability of losing. We lose if 0 or 1 of the 3 cards are hearts. So the probability of winning is $1 - \frac{\binom{39}{2}\binom{13}{1}}{\binom{52}{3}} - \frac{\binom{39}{3}}{\binom{52}{3}} = 64/425 = 0.1506$.

b. Put the cards in a row (e.g., before you look at them). Let X_1, X_2, X_3 indicate whether the first, second, third card are hearts. The total number of hearts is $X_1 + X_2 + X_3$. Also $\mathbb{E}(X_j) = 1/4$ for each j. So the expected number of hearts is $\mathbb{E}(X_1 + X_2 + X_3) = 1/4 + 1/4 + 1/4 = 3/4$.

2. Socks.

a. There are $\binom{21+8+4}{6} = 1107568$ equally-likely ways to pick the socks. There are $\binom{21}{2}\binom{8}{2}\binom{4}{2} = 35280$ ways to get 2 of each color. So the probability is $35280/1107568 = 0.0319$.

b. All of the socks are white with probability $\binom{21}{6}/\binom{33}{6} = 969/19778$. All of the socks are black with probability $\binom{8}{6}/\binom{33}{6} = 1/39556$. These possibilities are disjoint. We cannot get 6 black socks. So the total probability of getting 6 socks of the same color is $969/19778 + 1/39556 = 1939/39556 = 0.0490$.

c.

socks	probability
2 white, 4 black	$\binom{21}{2}\binom{8}{4}/\binom{33}{6} = 525/39556$
2 white, 4 brown	$\binom{21}{2}\binom{4}{4}/\binom{33}{6} = 15/79112$
2 black, 4 white	$\binom{8}{2}\binom{21}{4}/\binom{33}{6} = 5985/39556$
2 black, 4 brown	$\binom{8}{2}\binom{4}{4}/\binom{33}{6} = 1/39556$
2 brown, 4 white	$\binom{4}{2}\binom{21}{4}/\binom{33}{6} = 2565/79112$
2 brown, 4 black	$\binom{4}{2}\binom{8}{4}/\binom{33}{6} = 15/39556$

The total probability (add the probabilities of these disjoint events) is $1954/9889 = 0.1976$.

3. Married couples.

Let X_j indicate if the jth man sits across from his wife. Then $\mathbb{E}(X_j) = 1/19$, since no matter where the man sits, his wife can sit in 19 other possible chairs, only 1 of which is directly across from him. Thus the expected number of men sitting across from their wives is $\mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10}) = 1/19 + \cdots + 1/19 = 10/19$.

4. Ramen Noodles.

a. We either need 1 beef and 2 chickens, which happens with probability $\frac{\binom{10}{1}\binom{10}{2}}{\binom{20}{3}} = 15/38$, or 2 beef and 1 chicken, which also happens with probability 15/38. So the probability of at least one of each is $15/38 + 15/38 = 15/19$.

b. This is the complementary probability, i.e., $1 - 15/19 = 4/19$. If you prefer, you can calculate this directly, because the probability of 3 beefs is $\binom{10}{3}/\binom{20}{3} = 2/19$, and the probability of 3 chickens is 2/19, so again, the probability of three of the same flavor is $2/19 + 2/19 = 4/19$.

5. Picking letters at random.

a. There are $26^5 = 11881376$ ways that they could pick the letters, and there are $26!/21! = (26)(25)(24)(23)(22) = 7893600$ ways that the choices are unique. So the probability that they are unique is $7893600/11881376 = 18975/28561 = 0.6644$.

b. As before, there are $26^5 = 11881376$ equally-likely ways that they can make their choices.

Method #1: The number of ways that these are unique is $26!/21! = (26)(25)(24)(23)(22) = 7893600$, and for each of these possibilities, only 1 out of every 5! are in increasing order. So the probability that they are in increasing order is $\frac{7893600/5!}{11881376} = 1265/228488 = 0.005536$.

Method #2: There are $\binom{26}{5}$ ways to choose the letters without repeats, and once they are chosen, there is only 1 way to put them in increasing order. So the probability is $\binom{26}{5}/11881376 = 1265/228488 = 0.005536$.