STUDENT VERSION

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Probability Theory

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Questions on *Probability Theory*

Section 2

1. If the probability of event X, $P(X)$, equals 0 means that event X will never occur, and $P(X) = 1$ means that event X will always occur, then what does $P(X) = 0.5$ mean?

2. Let X be an event outside of sample space S. What is the probability of X, $P(X)$?

3. Based on the third rule displayed in the video

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

if the events A_j are disjoint. How might the relationship be affected if the events A_j are not necessarily disjoint?

Section 3

4. If four fair coins are flipped, what is the probability that there will be exactly two tails?

5. Suppose two dice are rolled and X is the event that the sum of the two dice is 7. How many different outcomes are in X^c? What is $P(X^c)$?

6. On any given day, I buy a lottery ticket. Consider the sample space S that represents my outcome. Define E_1 as the event that I win the lottery and E_2 as the event that I do not win the lottery. Since the sample space S consists of the two events E_1 and E_2, then the probability of winning the lottery is $\frac{1}{2}$. Should I start buying lottery tickets or is there something wrong with my reasoning?

Section 4

7. Suppose two dice are rolled. Let A be the event that the first die is odd. What is $P(A)$? Let B be the event that the second die is 1 or 6. What is $P(B)$? Are A and B disjoint? Are A and B independent?

8. If events A and B are not disjoint, and events B and C are not disjoint, is it possible for events A and C to be disjoint?

9. If events A and B are not independent, and events B and C are not independent, is it possible for events A and C to be independent?

Section 5

10. Alice and Bob are playing a game with an unfair coin. Alice starts by flipping the coin. If she flips a heads, she wins. Otherwise, she passes the coin to Bob, who then flips the coin. If he flips a tails, he wins. Otherwise, he passes the coin back to Alice and they repeat this process until someone wins. If the probability that Alice wins is $\frac{1}{2}$, what is the probability that Alice wins on the first turn?

Section 6

11. Note that $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Since $P(B)$ is in the denominator, we know we are dividing by something potentially less than 1. What guarantees that $P(A|B) \leq 1$?

12. Suppose there is a bag containing 5 red marbles and 5 blue marbles. Three marbles are drawn, with replacement each time. Given that at least two of the marbles drawn were blue, what is the probability that all three marbles drawn were blue?

13. Suppose $P(A|B) = \frac{2}{3}$ and $P(B|C) = \frac{1}{2}$. What other quantities do we need to know to determine $P(A|C)$?

14. *[Monty Hall Problem]* In the American television game show *Let's Make a Deal*, the player is given the choice of three doors. Behind one of the doors is a car, and behind the other two are goats. The player picks a door, and the host, who knows what's behind each door, always opens a different door from the door chosen by the player and always reveals a goat by this action. In this case, let's say the player picks door number 1 and the host opens door number 2, revealing a goat. The host then asks if the player wants to switch to the other unopened door, in this case, door number 3. Is it to the player's advantage to switch the choice?

15. It is always to the player's advantage to switch the choice! The only way the player can win the car without switching is if the player initially picks the door with the car. The occurs with probability $\frac{1}{3}$ since there are three doors, only one of which leads to a car.

    On the other hand, if the player initially picks a door containing a goat, then the game host reveals the other door containing a goat. Hence, switching would guarantee the car with probability $\frac{2}{3}$. The player doubles the chances of winning a car! *[Boy or Girl Paradox]*

    (a) Suppose a family has two children. If the older child is a boy, what is the probability that both children are boys?

(b) Suppose a family has two children. At least one of the children is a girl. What is the probability that both children are girls? The answer is not the same as the previous part!

Section 7

16. Suppose $P(A) = P(B)$. What do we know about the relationship between $P(A|B)$ and $P(B|A)$?

17. Let S represent the set of all outcomes and A be a given event. We saw that $P(S|A) = 1$. What is $P(A|S)$?

18. Find a counterexample for the claim

$$P(A \cap B \cap C) = P(C)P(A|C)P(B|C).$$

Section 8

19. Suppose we draw a five card hand at random from a standard 52-card deck. Given that we draw exactly 1 red card, what is the probability that we drew the ace of spades? Solve the problem with and without Bayes' Theorem.

20. Suppose we have a bag with 5 red marbles and 5 blue marbles. If we draw three marbles, without replacement, and we know that at least two of the marbles are blue, what is the probability that all three marbles are blue? Solve the problem with and without Bayes' Theorem.

Section 9

21. Suppose we roll two dice. Let X represent the sum of the two dice. Determine X for each outcome in the sample space. Is X discrete or continuous?

22. Suppose we flip five coins. Let X represent the number of heads and Y represent the greatest number of consecutive heads flipped during the sequence. What is $P(X + Y = 8)$?

23. When might an indicator random variable be useful?

Section 10

24. What is a probability mass function? Why might it be useful?

25. Suppose we roll three dice. Let X be the sum of the three dice. Determine the values of X which have nonzero mass.

26. Suppose we flip a fair coin four times. Let H be the number of heads. Determine the probability mass function for H.

Section 11

27. Let F be a cumulative distribution function of a random variable X. If X has minimum value m and maximum value M and there exists real values a and b such that $0 < F(a) < F(b) < 1$, what do we know about the relationship of m, M, a, b? Let r and s be real values such that $r < m$ and $s > M$. What are $F(r)$ and $F(s)$?

28. Suppose we flip a fair coin four times. Let H Be the number of heads and F be the cumulative distribution function for H. Determine the values of F.

29. Suppose X is a continuous random variable and $g(X)$ is the probability mass function. If $f(X)$ is the cumulative distribution function, what is the relationship between $f(X)$ and $g(X)$?

Section 12

30. Suppose we flip a fair coin three times and then roll two dice. Let X be the number of heads and Y be the sum of the dice.
    (a) Let p be the joint probability mass function. What is $p_{X,Y}(3,7)$?
    (b) Let F be the joint cumulative distribution function. What is $F_{X,Y}(1,7)$?

Section 13

31. Suppose that X and Y be random variables which are functions of the outcomes of events A and B respectively. If A and B are not independent, can X and Y be independent?

32. Let X and Y be random variables and p and F be the probability mass and cumulative distributive functions, respectively. Prove that $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x, y if and only if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

for all x, y.

Section 14

33. Suppose we roll two dice. Let X be the random variable which represents the sum of the two dice and Y be the value of the second die. Determine the probability mass function at the following values:
    (a) $p_{X|Y}(7,y)$
    (b) $p_{Y|X}(y,7)$
    (c) $p_{X|Y}(x,3)$
    (d) $p_{Y|X}(3,x)$

Section 15

34. Let H be the random variable for the number of heads obtained in flipping a fair coin four times. Find the expected value of H^2.

35. Suppose X and Y are independent random variables. Prove that

$$E[XY] = E[X]E[Y].$$

36. Let X and Y be random variables such that $X \leq Y$ for any outcome. Prove that

$$E[X] \leq E[Y].$$

37. How might we extend the definition of expected value to a continuous random variable?

Section 16

38. (a) What is the expected number of flips that a fair coin must be tossed to obtain a heads?
    (b) What is the expected number of flips that a fair coin must be tossed to obtain two heads?
    (c) What is the expected number of flips that a fair coin must be tossed to obtain n heads?
    (d) What is the expected number of flips that a fair coin must be tossed to obtain two consecutive heads?

Section 17

39. Suppose two dice are repeatedly rolled. Find the expected number of rolls to obtain a sum of 7.

40. Provide a counterexample to the claim that, for all functions f and random variables X,

$$f(E[X]) = E[f(X)].$$

41. Determine the expected value of the sum of a roll of two dice, the expected value of a function of a random variable.

Section 18

42. Let X be a random letter chosen from the alphabet. Determine the entropy of X.

43. Suppose we have a biased coin which lands heads $\frac{3}{4}$ of the time, and tails the other $\frac{1}{4}$ of the time. Determine the entropy of a single flip of the coin.

44. Let Y be a random variable with n possible equally likely outcomes. Determine the entropy of Y.

Section 20

45. Suppose I have a biased coin that lands heads with probability p and tails with probability $1-p$. Determine the value of p which gives the greatest entropy for a single flip.

46. I have a weighted die that lands on 1 exactly $\frac{1}{9}$ of the time, lands on 2 exactly $\frac{1}{4}$ of the time, lands on 3 exactly $\frac{1}{6}$ of the time, lands on 4 exactly $\frac{1}{9}$ of the time, lands on 5 exactly $\frac{1}{4}$ of the time, and lands on 6 exactly $\frac{1}{9}$ of the time. What is the entropy of a single roll of the die?

Section 22

47. Let X be a random vowel chosen from the alphabet. Determine the entropy of X.

48. A strange alien language has an alphabet consisting of 64 letters. Determine the entropy of a random string of 10 letters in this alien language.

Section 23

49. Prove the linearity of expected value. That is, if X and Y are random variables, and α and β are constants, then

$$E[\alpha X+\beta Y]=\alpha E[X]+\beta E[Y].$$

50. If X is a random variable and α is a constant, prove

$$Var(\alpha X)=\alpha^2 Var(X),$$

$$Var(X+\alpha)=Var(X).$$

51. Let X be the random variable which represents one roll of a die. Find the variance of X.

Section 24

52. Find a counterexample to the claim that the variance of the sum of two random variables is the sum of the variances of the two random variables.

53. Let f, g be functions and X and Y be random variables. Prove

$$E[f(X)g(Y)]=E[f(X)]E[g(Y)].$$