INSTRUCTOR VERSION WITH ANSWERS & FEEDBACK SHOWN

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Probability Theory

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Questions on *Probability Theory*

Section 2

1. If the probability of event X, $P(X)$, equals 0 means that event X will never occur, and $P(X) = 1$ means that event X will always occur, then what does $P(X) = 0.5$ mean?

   Solution: $P(X) = 0.5$ means that the event X is expected to occur approximately half the time.

2. Let X be an event outside of sample space S. What is the probability of X, $P(X)$?

   Solution: Since X is outside the sample space S, then X must never occur. Hence, the probability of X, $P(X)$ must equal zero. Thus, $P(X) = 0$.

3. Based on the third rule displayed in the video

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

   if the events A_j are disjoint. How might the relationship be affected if the events A_j are not necessarily disjoint?

   Solution: If the events A_j are not necessarily disjoint, then there may be some overlap in results for which A_i and A_k both occur, with $i \neq k$. Thus,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n)$$

Section 3

4. If four fair coins are flipped, what is the probability that there will be exactly two tails?

   Solution: The sixteen possibilities are

$$HHHH, \quad HHHT, \quad HHTH, \quad HHTT$$
$$HHHH, \quad HTHT, \quad HTTH, \quad HTTT$$
$$HHHH, \quad THHT, \quad THTH, \quad THTT$$

$$HHHH, \qquad TTHT, \qquad TTTH, \qquad TTTT.$$

Of the 16 possibilities, six outcomes have exactly two tails. Hence, the probability is $\frac{6}{16} = \frac{3}{8}$.

5. Suppose two dice are rolled and X is the event that the sum of the two dice is 7. How many different outcomes are in X^c? What is $P(X^c)$?

   Solution: The six outcomes of X are

   $$(1,6) \qquad (2,5) \qquad (3,4) \qquad (4,3) \qquad (5,2) \qquad (6,1).$$

   Since there are 36 total possibilities, then there are $36 - 6 = 30$ different outcomes in X^c. Hence, $P(X^c) = \frac{30}{36} = \frac{5}{6}$.

6. On any given day, I buy a lottery ticket. Consider the sample space S that represents my outcome. Define E_1 as the event that I win the lottery and E_2 as the event that I do not win the lottery. Since the sample space S consists of the two events E_1 and E_2, then the probability of winning the lottery is $\frac{1}{2}$. Should I start buying lottery tickets or is there something wrong with my reasoning?

   Solution: Although there are two outcomes, the chance of each outcome occurring is massively different. This fact must be taken into account before determining probability. The chances of E_1 are somewhere around the ballpark of one in a million, whereas the chances of E_2 which is almost 1.

Section 4

7. Suppose two dice are rolled. Let A be the event that the first die is odd. What is $P(A)$? Let B be the event that the second die is 1 or 6. What is $P(B)$? Are A and B disjoint? Are A and B independent?

   Solution: Of the six equally likely outcomes for the first die, A occurs when a 1, 3, or 5 are rolled. Hence, $P(A) = \frac{1}{6}$. Of the six equally likely outcomes for the second die, B occurs when a 1 or 6 are rolled. Hence, $P(B) = \frac{2}{6} = \frac{1}{3}$.

   A and B both occur in several situations (for example, if both dice are 1) so A and B are not disjoint. On the other hand, any result from the first die will not affect the outcome of the second die, so A and B are independent.

8. If events A and B are not disjoint, and events B and C are not disjoint, is it possible for events A and C to be disjoint?

   Solution: Yes. For example, if we roll two dice, if A is the event that a 1 is rolled on the first die, B is the event that a 1 is rolled on the second die, and C is the event that a 2 is rolled on the first die, then A and B are not disjoint, B and C are not disjoint, but A and C are disjoint.

9. If events A and B are not independent, and events B and C are not independent, is it possible for events A and C to be independent?

   Solution: Yes. For example, if we roll two dice, if A is the event that a 1 is rolled on the first die, B is the event that a 1 is rolled on both dice, and C is the event that a 1 is rolled on the second die, then A and B are not independent, B and C are not independent, but A and C are independent.

Section 5

10. Alice and Bob are playing a game with an unfair coin. Alice starts by flipping the coin. If she flips a heads, she wins. Otherwise, she passes the coin to Bob, who then flips the coin. If he flips a tails, he wins. Otherwise,

he passes the coin back to Alice and they repeat this process until someone wins. If the probability that Alice wins is $\frac{1}{2}$, what is the probability that Alice wins on the first turn?

Solution: Let the probability that the unfair coin lands heads be p. Then the probability that Alice wins on her first turn is p. Alternatively, if Alice wins on her second turn, then she flipped a tails on the first turn, a heads on her second turn, and in between, Bob also flipped a heads. Hence, the probability is $(1-p)p^2$. Continuing this reasoning, the probability that Alice wins is

$$p + (1-p)p^2 + (1-p)^2 p^3 + \ldots = \frac{1}{2}.$$

Hence, we have

$$\frac{p}{1 - p(1-p)} = \frac{1}{2}.$$

Solving the quadratic, we obtain

$$p = \frac{3 \pm \sqrt{5}}{2}.$$

However, p is a probability, which means that it must be between 0 and 1. Thus, the probability of the unfair coin landing heads (which is also the probability that Alice wins on her first turn) is

$$p = \frac{3 - \sqrt{5}}{2}.$$

Section 6

11. Note that $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Since $P(B)$ is in the denominator, we know we are dividing by something potentially less than 1. What guarantees that $P(A|B) \leq 1$?

    Solution: Since $A \cap B$ can only occur if event B occurs, we know that $P(A \cap B) \leq P(B)$. Hence $P(A|B) \leq 1$.

12. Suppose there is a bag containing 5 red marbles and 5 blue marbles. Three marbles are drawn, with replacement each time. Given that at least two of the marbles drawn were blue, what is the probability that all three marbles drawn were blue?

    Solution: The eight outcomes are

    $$\begin{array}{cccc} RRR & RRB & RBR & RBB \\ BRR & BRB & BBR & BBB. \end{array}$$

    Of the above outcomes, four outcomes have at least two of the marbles blue, and one outcome has all three marbles blue. Hence, the probability is $\frac{1}{4}$.

13. Suppose $P(A|B) = \frac{2}{3}$ and $P(B|C) = \frac{1}{2}$. What other quantities do we need to know to determine $P(A|C)$?

    Solution: Given that C occurs, event B must either occur or not occur. If event B occurs, we know the probability that A will occur. Hence, we need the quantities $P(A|B^c)$ and $P(B^c|C)$ before we can determine $P(A|C)$.

14. *[Monty Hall Problem]* In the American television game show *Let's Make a Deal*, the player is given the choice of three doors. Behind one of the doors is a car, and behind the other two are goats. The player picks a door, and the host, who knows what's behind each door, always opens a different door from the door chosen by the player and always reveals a goat by this action. In this case, let's say the player picks door number 1 and the host opens door number 2, revealing a goat. The host then asks if the player wants to switch to the other unopened door, in this case, door number 3. Is it to the player's advantage to switch the choice?

    Solution:

15. It is always to the player's advantage to switch the choice! The only way the player can win the car without switching is if the player initially picks the door with the car. The occurs with probability $\frac{1}{3}$ since there are three doors, only one of which leads to a car.

    On the other hand, if the player initially picks a door containing a goat, then the game host reveals the other door containing a goat. Hence, switching would guarantee the car with probability $\frac{2}{3}$. The player doubles the chances of winning a car! *[Boy or Girl Paradox]*

    (a) Suppose a family has two children. If the older child is a boy, what is the probability that both children are boys?

    (b) Suppose a family has two children. At least one of the children is a girl. What is the probability that both children are girls? The answer is not the same as the previous part!

    (a) **Solution:** If the older child is a boy, the younger child can be either a girl or a boy, so the probability that both children are boys is $\frac{1}{2}$.

    (b) **Solution:** If a family has two children, the possible outcomes are

    $$GG, \qquad GB, \qquad BG, \qquad BB.$$

    Of the above, three outcomes have at least one children being a girl, so the probability that both children are girls is $\frac{1}{3}$.

Section 7

16. Suppose $P(A) = P(B)$. What do we know about the relationship between $P(A|B)$ and $P(B|A)$?

    Solution: We have
    $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A).$$

17. Let S represent the set of all outcomes and A be a given event. We saw that $P(S|A) = 1$. What is $P(A|S)$?

    Solution: Since S is the set of all outcomes, S always occurs. Hence, $P(A|S) = P(A)$.

18. Find a counterexample for the claim
    $$P(A \cap B \cap C) = P(C)P(A|C)P(B|C).$$

    Solution: If A and B are two events that can never occur together, then $P(A \cap B \cap C) = 0$, but $P(C)$, $P(A|C)$ and $P(B|C)$ may all be nonzero.

Section 8

19. Suppose we draw a five card hand at random from a standard 52-card deck. Given that we draw exactly 1 red card, what is the probability that we drew the ace of spades? Solve the problem with and without Bayes' Theorem.

    Solution: There are 26 red cards we could have drawn, and $\binom{26}{4}$ ways to choose 4 black cards from the remaining 26 cards. If we drew the ace of spades with exactly 1 red card, there are 26 possibilities for the red card and $\binom{25}{3}$ was to choose the other 3 black cards. Hence, the probability is
    $$\frac{\binom{25}{3}}{\binom{26}{4}}.$$

20. Suppose we have a bag with 5 red marbles and 5 blue marbles. If we draw three marbles, without replacement, and we know that at least two of the marbles are blue, what is the probability that all three marbles are blue? Solve the problem with and without Bayes' Theorem.

Solution: The probability that all three marbles is blue is

$$p_1 = \left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right).$$

The probability that at least two of the marbles is blue is

$$p_2 = \left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right) + 3\left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{5}{8}\right).$$

The first term in the summand is the probability all three marbles is blue and the second term in the summand is the probability that exactly two marbles are two. Since the two events are disjoint, then the probability that at least two of the marbles are blue is their sum. The result follows as $\frac{p_1}{p_2}$.

Section 9

21. Suppose we roll two dice. Let X represent the sum of the two dice. Determine X for each outcome in the sample space. Is X discrete or continuous?

Solution: For each of the 36 possible outcomes (m, n), $X = m + n$. Moreover, X can only obtain integer values, so X is discrete.

22. Suppose we flip five coins. Let X represent the number of heads and Y represent the greatest number of consecutive heads flipped during the sequence. What is $P(X + Y = 8)$?

Solution: The only possibility for $X + Y = 8$ is if $X = Y = 4$. The only sequences for which $X = Y = 4$ are

$$HHHHT \qquad THHHH.$$

Hence, $P(X + Y = 8) = \frac{2}{32} = \frac{1}{16}$.

23. When might an indicator random variable be useful?

Solution: One situation an indicator random variable might be useful is if we don't care about the result of a particular event, just whether or not the event occurred.

Section 10

24. What is a probability mass function? Why might it be useful?

Solution: A probability mass function gives the probability of obtaining specific values in a given range. It is useful to see which possibilities have nonzero probability and which possibilities are impossible.

25. Suppose we roll three dice. Let X be the sum of the three dice. Determine the values of X which have nonzero mass.

Solution: If all three dice come up with one, then the sum is 3, whereas the maximum sum of 18 occurs when all three dice result in six. Hence, the values of X which have nonzero mass are all integers between 3 and 18, inclusive.

26. Suppose we flip a fair coin four times. Let H be the number of heads. Determine the probability mass function for H.

Solution: The sixteen possibilities are

$$HHHH, \quad HHHT, \quad HHTH, \quad HHTT$$

$$HHHH, \quad HTHT, \quad HTTH, \quad HTTT$$
$$HHHH, \quad THHT, \quad THTH, \quad THTT$$
$$HHHH, \quad TTHT, \quad TTTH, \quad TTTT.$$

Hence,

$$\begin{aligned} H &= 0 \qquad p = \frac{1}{16} \\ H &= 1 \qquad p = \frac{4}{16} = \frac{1}{4} \\ H &= 2 \qquad p = \frac{6}{16} = \frac{3}{8} \\ H &= 3 \qquad p = \frac{4}{16} = \frac{1}{4} \\ H &= 4 \qquad p = \frac{1}{16} \end{aligned}$$

Section 11

27. Let F be a cumulative distribution function of a random variable X. If X has minimum value m and maximum value M and there exists real values a and b such that $0 < F(a) < F(b) < 1$, what do we know about the relationship of m, M, a, b? Let r and s be real values such that $r < m$ and $s > M$. What are $F(r)$ and $F(s)$?

Solution: Since m is the minimum value, then if $a < m$, we would have $F(a) = 0$, as no values below m have any mass. Hence, $a \geq m$. By similar reasoning,

$$m \leq a < b \leq M.$$

Moreover, since no values below m have any mass and $r < m$, then $F(r) = 0$. Similarly, $F(s) = 1$.

28. Suppose we flip a fair coin four times. Let H Be the number of heads and F be the cumulative distribution function for H. Determine the values of F.

Solution: From a previous problem, we have

$$\begin{aligned} H &= 0 \qquad p = \frac{1}{16} \\ H &= 1 \qquad p = \frac{4}{16} = \frac{1}{4} \\ H &= 2 \qquad p = \frac{6}{16} = \frac{3}{8} \\ H &= 3 \qquad p = \frac{4}{16} = \frac{1}{4} \\ H &= 4 \qquad p = \frac{1}{16}. \end{aligned}$$

Thus,

$$\begin{aligned} F&=0 \qquad p=\frac{1}{16}\\ F&=1 \qquad p=\frac{5}{16}\\ F&=2 \qquad p=\frac{11}{16}\\ F&=3 \qquad p=\frac{15}{16}\\ F&=4 \qquad p=1. \end{aligned}$$

29. Suppose X is a continuous random variable and $g(X)$ is the probability mass function. If $f(X)$ is the cumulative distribution function, what is the relationship between $f(X)$ and $g(X)$?

Solution: Notice that

$$f(a)=\sum_{i=-\infty}^{a} g(a)$$

when X is a discrete random variable. Hence,

$$f(a)=\int_{-\infty}^{a} g(x)\,dx,$$

when X is a continuous random variable.

Section 12

30. Suppose we flip a fair coin three times and then roll two dice. Let X be the number of heads and Y be the sum of the dice.

(a) Let p be the joint probability mass function. What is $p_{X,Y}(3,7)$?

(b) Let F be the joint cumulative distribution function. What is $F_{X,Y}(1,7)$?

Solution:

(a) The probability that $X=3$ is $\frac{1}{8}$ since all three flips must come up heads. The probability that $Y=7$ is $\frac{1}{6}$. Hence, $p_{X,Y}(3,7)=\frac{1}{48}$.

(b) Notice that the outcomes of X do not affect the outcomes of Y, so we can consider their individual outcomes. The probability of $X=0$ is $\frac{1}{8}$. The probability of $X=1$ is $\frac{3}{8}$. Hence, the probability of $X\leq 1$ is $\frac{1}{2}$. The probability of getting $Y\leq 7$ is

$$\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}+\frac{6}{36}=\frac{7}{12}.$$

Hence, $F_{X,Y}(1,7)=\frac{7}{24}$.

Section 13

31. Suppose that X and Y be random variables which are functions of the outcomes of events A and B respectively. If A and B are not independent, can X and Y be independent?

Solution: No. If specific outcomes of A and B are dependent, then so are the corresponding values of X and Y assigned to those outcomes.

32. Let X and Y be random variables and p and F be the probability mass and cumulative distributive functions, respectively. Prove that $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x, y if and only if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

for all x, y.

Solution: Note that

$$\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y) \\ \frac{d}{dt}\left(F_{X,Y}(x,y)\right) &= \frac{d}{dt}\left(F_X(x)F_Y(y)\right) \\ p_{X,Y}(x,y) &= p_X(x)p_Y(y) \end{aligned}$$

Additionally, $p_{X,Y}(m_1, m_2) = p_X(m_1)p_Y(m_2)$ for the minimum values of X and Y at m_1 and m_2, respectively. Hence, the initial values conditions are satisfied for both sides, so the steps are reversible and the result follows.

Section 14

33. Suppose we roll two dice. Let X be the random variable which represents the sum of the two dice and Y be the value of the second die. Determine the probability mass function at the following values:

(a) $p_{X|Y}(7, y)$
(b) $p_{Y|X}(y, 7)$
(c) $p_{X|Y}(x, 3)$
(d) $p_{Y|X}(3, x)$

(a) **Solution:** Regardless of the value of the second die, the probability that the sum is 7 is $\frac{1}{6}$, so $p_{X|Y}(7, y) = \frac{1}{6}$.

(b) **Solution:** If the sum of the dice is 7, the value of the second die can still be any value with equal probability. Hence, $p_{X|Y}(y, 7) = \frac{1}{6}$

(c) **Solution:** If the value of the second die is 3, then the possible sums are the integer values from 4 to 9, each of which have probability $\frac{1}{6}$.

(d) **Solution:** If the sum of the two dice is 3, then the possible outcomes are $(1, 2)$ and $(2, 1)$. Hence, the possibilities of the second die are 1 and 2, each with probability $\frac{1}{2}$. All other values of $p_{Y|X}(3, x)$ are zero.

Section 15

34. Let H be the random variable for the number of heads obtained in flipping a fair coin four times. Find the expected value of H^2.

Solution:

$$E[H^2] = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}.$$

35. Suppose X and Y are independent random variables. Prove that

$$E[XY] = E[X]E[Y].$$

Solution:

$$\begin{aligned} E[XY] &= \sum p(X = x, Y = y) \cdot X(x)Y(y) \\ &= \sum p(X = x)p(Y = y) \cdot X(x)Y(y) \\ &= \sum (p(X = x) \cdot X(x))(p(Y = y) \cdot Y(y)) \end{aligned}$$

Notice that the sum is over all values for which X or Y have mass. However, for the values for which X has mass, Y is independent and can be factored outside of the sum. Similarly, X can be factored outside the sum for the values for which Y has mass. Therefore,

$$E[XY] = \sum (p(X = x) \cdot X(x)) \sum ((p(Y = y) \cdot Y(y)) = E[X]E[Y].$$

36. Let X and Y be random variables such that $X \leq Y$ for any outcome. Prove that

$$E[X] \leq E[Y].$$

Solution: Consider the definition of expected value over all possible values for which X and Y have mass. Then

$$E[X] = \sum p(X = n)X(n) \leq \sum p(Y = n)Y(n) = E[Y],$$

and as a result, $E[X] \leq E[Y]$.

37. How might we extend the definition of expected value to a continuous random variable?

Solution: Since the expected value is the sum of discrete values, we would expect to take an integral over continuous random variables. That is,

$$E[X] = \int p(t)X(t)\, dt,$$

where p serves the same function as the probability mass function, but is instead called the probability density function as an extension to continuous random variables.

Section 16

38. (a) What is the expected number of flips that a fair coin must be tossed to obtain a heads?
    (b) What is the expected number of flips that a fair coin must be tossed to obtain two heads?
    (c) What is the expected number of flips that a fair coin must be tossed to obtain n heads?
    (d) What is the expected number of flips that a fair coin must be tossed to obtain two consecutive heads?

    (a) **Solution:** The probability of heads is $\frac{1}{2}$, so the expected number of flips to obtain a heads is 2.

    (b) **Solution:** From the previous part, two flips are expected to obtain a heads. Hence, 4 flips are expected to obtain two heads.

    (c) **Solution:** Since two flips are expected to obtain a heads, then $2n$ flips are expected to obtain n heads.

(d) **Solution:** Let E be the expected number of flips that a fair coin must be tossed to obtain two consecutive heads. The first flip is tails with probability $\frac{1}{2}$, in which case E more flips are expected. If the first flip is heads, and the second flip is tails, E more flips are expected after the first two flips. This occurs with probability $\frac{1}{4}$. Finally, if both flips are heads, with probability $\frac{1}{4}$, then only two flips are necessary. Thus, E satisfies the relationship

$$E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{4} \cdot 2,$$

so the expected number of flips is $E = 6$.

Section 17

39. Suppose two dice are repeatedly rolled. Find the expected number of rolls to obtain a sum of 7.

Solution: The probability of rolling a sum of 7 is $\frac{1}{6}$. Hence, the expected number of rolls is 6.

40. Provide a counterexample to the claim that, for all functions f and random variables X,

$$f(E[X]) = E[f(X)].$$

Solution: Let $f(x) = x^2$ and X be a random variable which equals 0 with probability $p = \frac{1}{2}$ and equals 1 with probability $p = \frac{1}{2}$. Then $f(E[X]) = \frac{1}{4}$, but $E[f(x)] = \frac{1}{2}$, so the claim is not true.

41. Determine the expected value of the sum of a roll of two dice, the expected value of a function of a random variable.

Solution: Since the two dice are independent, the expected value of the sum is just the sum of the expected value, by linearity of expectation. The expected value of a single die is $\frac{7}{2}$, so the expected value of the sum of a roll of two dice is simply 7.

Section 18

42. Let X be a random letter chosen from the alphabet. Determine the entropy of X.

Solution: Since each letter has probability $\frac{1}{26}$ of being chosen, the entropy of X is simply $\log 26$.

43. Suppose we have a biased coin which lands heads $\frac{3}{4}$ of the time, and tails the other $\frac{1}{4}$ of the time. Determine the entropy of a single flip of the coin.

Solution: The entropy is

$$-\frac{1}{4}\log\left(\frac{1}{4}\right) - \frac{3}{4}\log\left(\frac{3}{4}\right).$$

44. Let Y be a random variable with n possible equally likely outcomes. Determine the entropy of Y.

Solution: Since each outcome has probability $\frac{1}{n}$ of being chosen, the entropy is

$$\sum_{i=1}^{n} \frac{1}{n}\log(n) = \log(n).$$

Section 20

45. Suppose I have a biased coin that lands heads with probability p and tails with probability $1 - p$. Determine the value of p which gives the greatest entropy for a single flip.

Solution: The entropy is

$$-p \log p - (1-p)\log(1-p),$$

which has derivative

$$-\log(p) + \log(1-p),$$

which equals zero at $p = \frac{1}{2}$. Additionally, this value is a local maximum. Hence, the value of p which gives the greatest entropy for a single flip is $p = \frac{1}{2}$. Intuitively, the greatest amount of entropy for a single flip occurs with the greatest amount of uncertainty, which happens when both outcomes are equally likely, with $p = \frac{1}{2}$.

46. I have a weighted die that lands on 1 exactly $\frac{1}{9}$ of the time, lands on 2 exactly $\frac{1}{4}$ of the time, lands on 3 exactly $\frac{1}{6}$ of the time, lands on 4 exactly $\frac{1}{9}$ of the time, lands on 5 exactly $\frac{1}{4}$ of the time, and lands on 6 exactly $\frac{1}{9}$ of the time. What is the entropy of a single roll of the die?

Solution: The entropy is

$$\frac{1}{9}\log(9) + \frac{1}{4}\log(4) + \frac{1}{6}\log(6) + \frac{1}{9}\log(9) + \frac{1}{4}\log(4) + \frac{1}{9}\log(9).$$

Section 22

47. Let X be a random vowel chosen from the alphabet. Determine the entropy of X.

Solution: Let n be the number of vowels in the alphabet. Depending on your interpretation of the definition of vowel, n could be five, six, or some other value. However you choose to interpret the definition, $\log n$ is the entropy of X because each outcome has equal probability.

48. A strange alien language has an alphabet consisting of 64 letters. Determine the entropy of a random string of 10 letters in this alien language.

Solution: Each character has $\log(64)$ entropy. Since each character of the string is independent, then the total entropy is $10\log(64)$.

Section 23

49. Prove the linearity of expected value. That is, if X and Y are random variables, and α and β are constants, then

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y].$$

Solution: Considering the sum over all values which have mass,

$$\begin{aligned} E[\alpha X + \beta Y] &= \sum p(n)(\alpha \cdot X(n) + \beta Y(n)) \\ &= \alpha \sum p(n) \cdot X(n) + \beta \sum p(n) \cdot Y(n) \\ &= \alpha E[X] + \beta E[Y] \end{aligned}$$

50. If X is a random variable and α is a constant, prove

$$Var(\alpha X) = \alpha^2 Var(X),$$

$$Var(X + \alpha) = Var(X).$$

Solution: By the definition of variation,

$$\begin{aligned} Var(\alpha X) &= E[\alpha^2 X^2] - (E[\alpha X])^2 \\ &= \alpha^2 (E[X^2] - (E[X])^2) \\ &= \alpha^2 Var(X). \end{aligned}$$

$$\begin{aligned} Var(X + \alpha) &= E[X^2 + 2\alpha X + \alpha^2] - (E[X + \alpha])^2 \\ &= E[X^2] + 2\alpha E[X] + \alpha^2 - (E[X] + \alpha)^2 \\ &= E[X^2] + 2\alpha E[X] + \alpha^2 - (E[X])^2 - 2\alpha E[X] - \alpha^2 \\ &= E[X^2] - (E[X])^2 \\ &= Var(X). \end{aligned}$$

51. Let X be the random variable which represents one roll of a die. Find the variance of X.

Solution: As determined in a previous problem, $E[X^2] = \frac{91}{6}$ and $E[X] = \frac{7}{2}$. Hence, the variance is $\frac{91}{6} - \frac{49}{4} = \frac{35}{12}$.

Section 24

52. Find a counterexample to the claim that the variance of the sum of two random variables is the sum of the variances of the two random variables.

Solution: Let A and B be random variables such that $A = 0$ with probability $p = \frac{1}{2}$ and $A = 1$ with probability $p = \frac{1}{2}$. Let $A = B$. Then the variance of A is $\frac{1}{4}$, so the sum of the variances of the two random variables is $\frac{1}{2}$. On the other hand, $A + B = 0$ half the time and 2 the other half of the time, so the variance of the sum of the two random variables is 1. Therefore, the claim is not true.

53. Let f, g be functions and X and Y be random variables. Prove

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)].$$

Solution: Let A and B be random variables such that $A = f(X)$ and $B = g(Y)$. Then $E[AB] = E[A]E[B]$ from a previous problem, and the desired result follows.