STUDENT VERSION

June 6, 2017

Mathematical Theory of Communication

Center for Science of Information, A National Science Foundation Science & Technology Center http://soihub.org

Questions on *Mathematical Theory of Communication*

Section 1: Coding

1. In the corresponding video, the messages "0", "10", "11" were assigned for sunny, rainy, and moderate, respectively. Why was the message "1" not assigned?

2. Suppose a message has four possibilities and we encode them as "00", "01", "10", "11". Why might this not be the most efficient scheme? Recall that we have not taken into account the probability of each message.

Section 2: Entropy with Variance

3. Suppose message M consists of L words, each of which could be n different possibilities. What is the maximum entropy message M could contain?

4. Suppose there are n outcomes to a random process. If this random process occurs once, and the outcome is given, what are possible conditions so that the entropy of the process is NOT $\log_2(n)$?

Section 3: Huffman Coding

5. Suppose we had a fixed length encoding scheme for 7 outcomes. How many bits would be needed? What is the average length of the encoding scheme?

6. Now suppose we are given the following distribution for 7 outcomes. Determine a Huffman encoding scheme

for the outcomes:

Outcome	Probability
A	$\frac{1}{8}$
B	$\frac{1}{4}$
C	$\frac{1}{16}$
D	$\frac{1}{16}$
E	$\frac{1}{8}$
F	$\frac{1}{8}$
G	$\frac{1}{4}$

7. What is the average length of the Huffman encoding scheme? How does it compare to the fixed length encoding scheme?

8. What is the entropy of the distribution? How does it compare to the average length of the Huffman encoding scheme? Is it possible for the Huffman encoding scheme to do better?

Section 4: Shannon's First Theorem

9. Suppose we are given a distribution in which 0 shows up with probability $p = 0.8$ and 1 shows up with probability $p = 0.2$. Determine a Huffman encoding scheme for the outcomes and compute the difference between the average length of the encoding scheme and the entropy of the distribution.

10. Now, group outcomes in series of two, so that there are now 4 possible outcomes. Determine a Huffman encoding scheme for the outcomes and compute the difference between the average length of the encoding scheme and the entropy of the distribution.

11. How does the difference in the first scheme compare to the difference in the second scheme? This is the idea behind Shannon's First Theorem!

Section 5: Kraft's Inequality

12. Determine a Huffman encoding scheme for the following distribution on 4 outcomes:

Outcome	Probability
A	$\frac{1}{12}$
B	$\frac{1}{3}$
C	$\frac{1}{12}$
D	$\frac{1}{2}$

13. Confirm Kraft's Inequality for the previous problem.

14. Prove the following lengths in bits for the encoding of each of the following 9 possible outcomes as instantaneous codes is not uniquely decodable given a binary alphabet.

Outcome	Length in Bits of Encoding
A	3
B	4
C	2
D	5
E	6
F	3
G	3
H	2
I	5

Section 6: Channels

15. What problems might a lossy channel present? Although it may not be possible to guarantee correctness, what are some ways we could increase accuracy of the encoding and decoding process?

16. Let X and Y be random variables representing a roll of a die such that Y is the result of the die and $X = 0$ if $Y = 6$ and $X = 1$ otherwise. What is the entropy of Y given X, $H(Y|X)$? What is the entropy of X given Y, $H(X|Y)$?

Section 7: Conditional and Joint Entropy

17. A cruel and unusual teacher distributes grades according to the flip of a coin. Suppose X is a random variable which represents the outcome of the coin. That is, $X = 1$ if the coin is heads and $X = 0$ if the coin is tails. Now, suppose Y is a random variable representing the grade given by the teacher. If the coin is heads, $Y = 100$ with probability $p = 0.5$, $Y = 80$ with probability $p = 0.25$ and $Y = 60$ with probability $p = 0.25$. On the other hand, if the coin is tails, $Y = 100$ with probability 0, $Y = 80$ with probability $p = \frac{1}{3}$ and $Y = 60$ with probability $p = \frac{2}{3}$.

    (a) Determine the conditional entropy of Y given X, $H(Y|X)$.

    (b) Determine the conditional entropy of X given Y, $H(X|Y)$.

    (c) Determine the joint entropy $H(X, Y)$ and confirm that

    $$H(X, Y) = H(Y|X) + H(X) = H(X|Y) + H(Y).$$