Investigating Symmetries of Uniform Attachment Graphs

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Introduction

The asymptotic behavior of the automorphism group of a random graph source as a function of graph size, in addition to being interesting in its own right as a combinatorial problem, has important information theoretic implications: a graph source that produces graphs with small automorphism groups has a corresponding structure source that is more compressible than the graph source.
Here we study, in particular, the behavior of the automorphisms of uniform attachment graphs. We prove symmetry for the single choice case and show progress toward a proof of asymmetry for the multiple choice case. In addition, we define a natural, related model that exhibits extremely different symmetry behavior. We also give numerical evidence for some of our claims.

Uniform Attachment Model: Definition

The uniform attachment model (a special case of a preferential attachment model, which is defined in, e.g., [1]) with parameters n and m (denoted by $G_{n,m}$) is defined as follows:

- $G_{1,m}$ consists of a single vertex and no edges.
- $G_{n,m}$ is generated from $G_{n-1,m}$ by adding a new vertex n and making m choices, uniformly at random with replacement, from $[n-1]$. For any $i \in [n-1]$, the edge (i, n) is added if and only if i is chosen at least once.

Symmetry Concepts

Definition (Automorphism)

Given a graph G on n vertices, a permutation $\pi : V(G) \to V(G)$ is said to be an automorphism if, for all $x, y \in V(G)$, $(x, y) \in E(G)$ if and only if $(\pi(x), \pi(y)) \in E(G)$. The set of automorphisms of G forms a group under function composition, and it is denoted by

$$\mathrm{Aut}(G).$$

Definition (Well behaved permutation sequence)

A sequence of permutations $\{\pi_n : [n] \to [n]\}_{n \geq n_0}$ is said to be *well behaved* if, for all $\alpha \in (0, 1]$, the limit

$$\lim_{n\to\infty} \frac{\pi_n(\lceil \alpha n \rceil)}{n}$$

exists.

From here on, when we speak of limits as $n \to \infty$ involving some permutation π, we assume that π is a well behaved sequence, and we are referring to π_n.

Problem Statement

The main problem is the following: prove or disprove

Claim

For $m = 1$,

$$\lim_{n\to\infty} \Pr[|\mathrm{Aut}(G_{n,m})| > 1] = 1,$$

and for $m \geq 2$, $m = \mathcal{O}_n(1)$,

$$\lim_{n\to\infty} \Pr[|\mathrm{Aut}(G_{n,m})| > 1] = 0.$$

$m = 1$ Case: Proof Sketch

First, note that an $m = 1$ uniform attachment graph is precisely a random recursive tree, which is well studied. Define a "cherry" to be a pair of leaves with the same parent. It suffices to prove that, with high probability, a uniform attachment tree has at least one cherry, for then it has as an automorphism that swaps the two leaves and leaves all other nodes fixed. We do this as follows:

- Invoke the result of Feng & Mahmoud [2] which implies that the number of cherries in a graph of size n, when appropriately normalized, asymptotically has a Gaussian distribution with mean $\Theta(n)$ and sufficiently small variance.
- An elementary argument then yields the desired result.

$m \geq 2$ Case: Proof Ideas

For the $m \geq 2$ case, we use a measure of asymmetry used previously in [3]. Given a graph G, a permutation $\pi : V(G) \to V(G)$, and a vertex $u \in V(G)$, we define the *defect* of u with respect to π as

$$D_\pi(u) = |N(\pi(u))\Delta\pi(N(u))|,$$

where $N(x)$ denotes the set of vertices adjacent to x and Δ denotes the symmetric difference of two sets. Define $D_\pi(G)$ to be

$$D_\pi(G) = \max_{u \in V(G)} D_\pi(u),$$

and define $D(G)$ as

$$\min_{\pi \neq \mathrm{ID}} D_\pi(G).$$

The defect has the property that $D_\pi(G) = 0$ if and only if $\pi \in \mathrm{Aut}(G)$, and $D(G) = 0$ if and only if $|\mathrm{Aut}(G)| > 1$. The core of the proof method, then, is to show that the defect of a graph is bounded away from 0 with high probability. Our results so far along these lines follow.

Proposition

Let $G \sim G_{n,m}$ *for* $m \geq 2$, $m = \mathcal{O}(1)$, *let* π *be a permutation of the vertices of* G, *let* $u \in V(G)$, *and define* $\omega(\pi, u) = \min\{u, \pi(u)\}$. *Then*

$$\mathrm{E}[D_\pi(u)] = \Theta\left(\log\left(\frac{n}{\omega(\pi, u)}\right)\right).$$

Proposition

Let $G \sim G_{n,m}$, *and let* π *be a permutation of the vertices of* G. *Then*

$$\lim_{n\to\infty} \Pr[\pi \in \mathrm{Aut}(G)] = 0.$$

Sliding Window Model

We proposed a natural generalization of the uniform and preferential attachment models that incorporates a window within which every new vertex is allowed to make connection choices. We assume that window sizes are independently distributed for all nodes, but they need not be identically distributed. It turns out that, for all $m \geq 1$, in both the uniform and preferential attachment cases, if the expected values of all window sizes are $\mathcal{O}(1)$ with respect to n, then graphs from this model are symmetric with high probability.

Numerical Evidence

Figure: Defect for $m = 3$

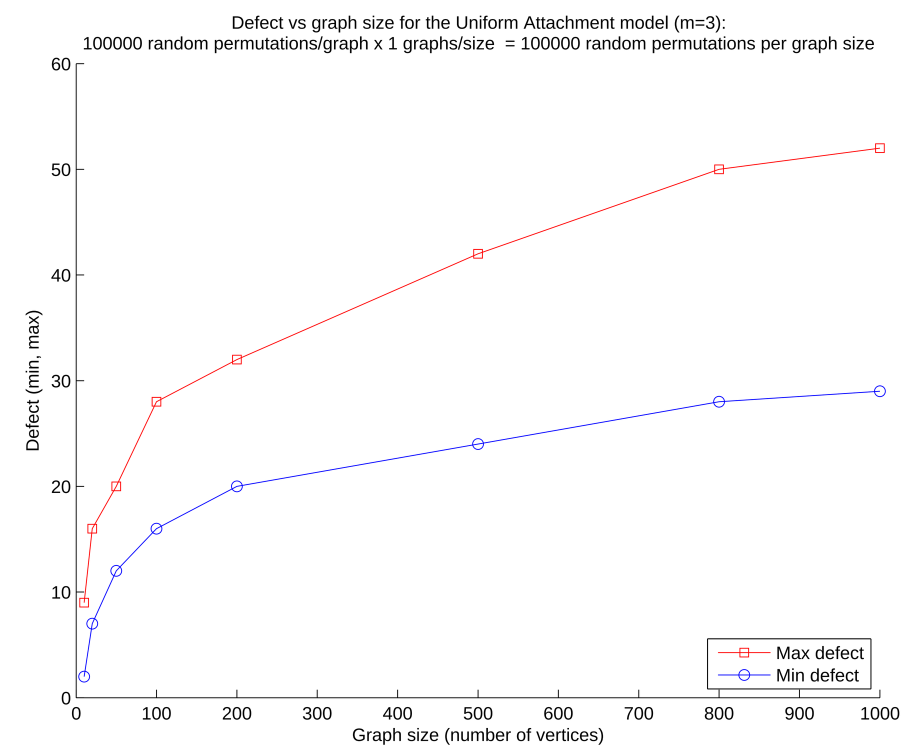
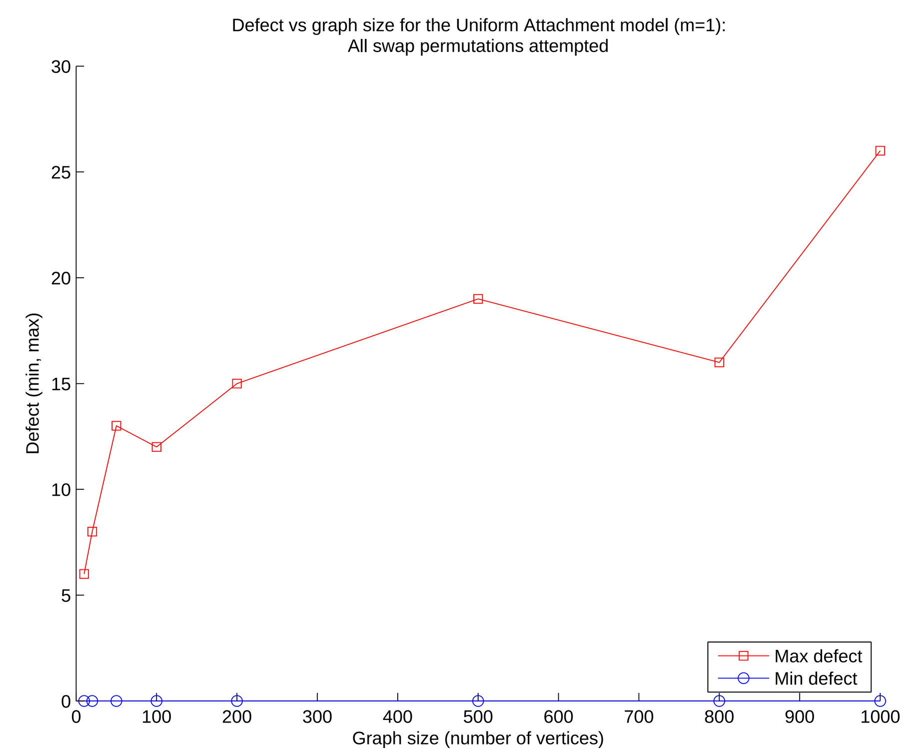


Figure: Defect for $m = 1$



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