Learning Algebraic Functions From a Few Samples

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Based on joint works with Amir Shpilka

April 28, 2014

Learning Long Division

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$$f(x, y) := x/y$$

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- $f(1, 1) = 1$

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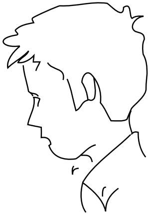

Examples:

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Claim

Now you know long division.

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$f(x, y) = x/y$ is the uniquely determined by the above,

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$f(x, y) = x/y$ is the uniquely determined by the above, as a function of the form $f(x, y) = (ax + b)/(cy + d)$. □

Theme

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Meta-fact

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- PAC (Probably Approximately Correct) learning
- Error-correcting Codes
- (Black-box) Polynomial Identity Testing (PIT) — given an algebraic circuit C, does the polynomial $C(\vec{x})$ equal zero?

More on the Theme

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Meta-hope

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- Coding theory: need **deterministic** coding schemes for communication
- Complexity theory: understanding the power of (pseudo)randomness
- Complexity theory: connections with circuit lower bounds

Bilinear Forms

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Question

Can we construct $\approx 2nr$ (resp. $4nr$) **explicit** *samples for identity testing (resp. learning)?*

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The number of measurements is optimal over algebraically closed fields. The use of rank-1 measurements is novel, and thus instantiates the meta-hope, by learning a function from deterministic evaluations.

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This is the *first* deterministic sub-exponential time algorithm for even determining if the d-linear form is non-zero,

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There is a deterministic $\mathrm{poly}(n, r, d)^{\log d}$*-time algorithm that reconstructs rank r tensors in $[n]^d$ from evaluations of their d-linear forms.*

This is the *first* deterministic sub-exponential time algorithm for even determining if the d-linear form is non-zero, by only using its evaluations.

Results — Low-Rank Recovery

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Any r-sparse-recovery oracle with measurements $\mathcal{V}$

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rank 1 measurements: do a clever change of basis from $\mathcal{H}$ to $\mathcal{H}'$

Sparsity and Rank (I)

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want: M has rank $\leq r$

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The rows with non-zeros amongst the entries a, b, c, d, e are linearly independent, as this is a triangular system, so this follows from standard linear algebra. So if the rank is at most 3, then this diagonal is 3-sparse.

□

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Given a row-reduced upper part of M, the next diagonal is essentially 3-sparse.

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Learning an 7×7, rank ≤ 3 matrix, essentially using 3-sparse recovery:

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Any r-sparse-recovery oracle with measurements $\mathcal{V}$ can be turned into a rank $\leq r$ low-rank recovery algorithm, with measurements $\mathcal{H}$, and $|\mathcal{H}| = 2n|\mathcal{V}|$.

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- called sparse-recovery $2n$ times — once per diagonal $\square$

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TOC