Information and inference on trees

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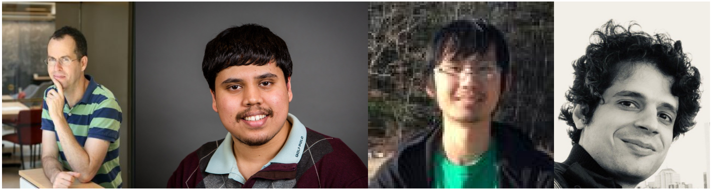
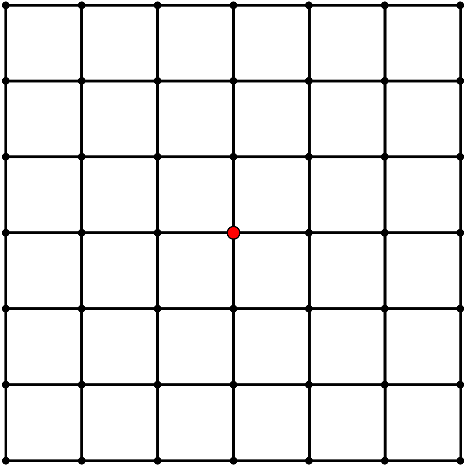


with: E. Mossel, A. Makur, Yuzhou Gu, H. Roozbehani

Motivation: Information Propagation in 2D Grid

- How does information spread in time?

$\boxed{\leftarrow\uparrow\downarrow\rightarrow \text{ are } \mathrm{BSC}(\delta)}$



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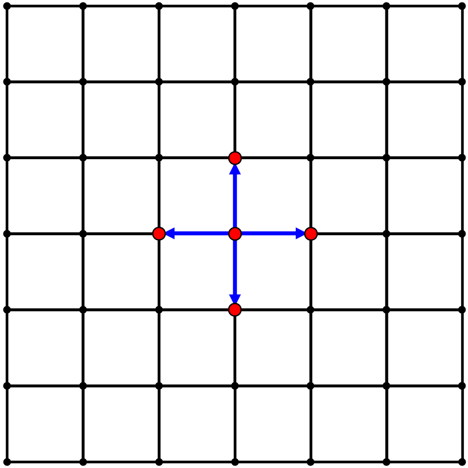

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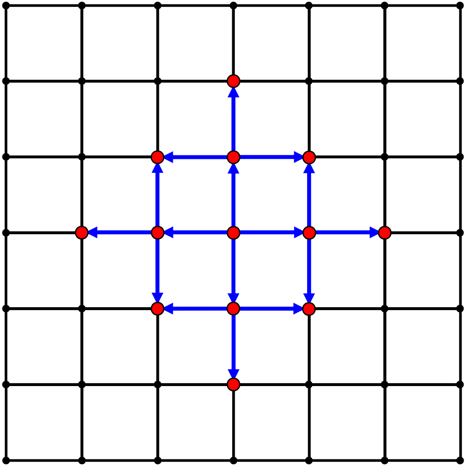

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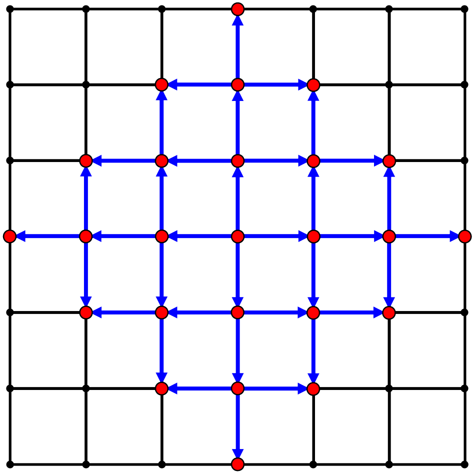

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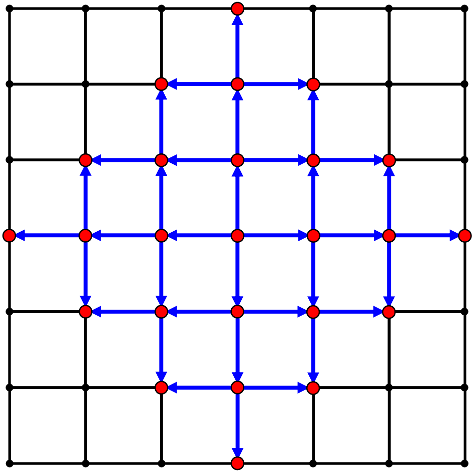
- Can we invent relay functions so that far boundary contains non-trivial information about the original bit?



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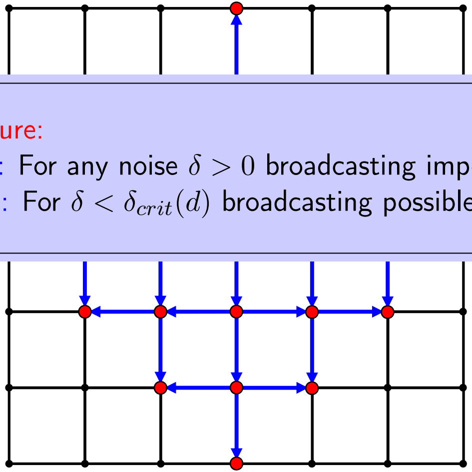

Main conjecture:

2 dimensions: For any noise $\delta > 0$ broadcasting impossible

$d \geq 3$ dimen.: For $\delta < \delta_{crit}(d)$ broadcasting possible

Related Models in the Literature

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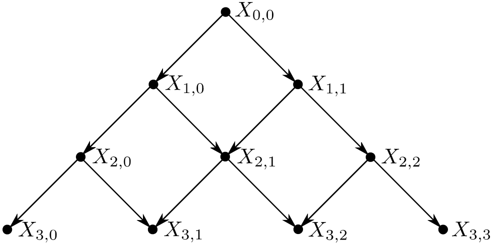
  Impossibility of broadcasting on 2D regular grid parallels ergodicity of 1D probabilistic cellular automata.



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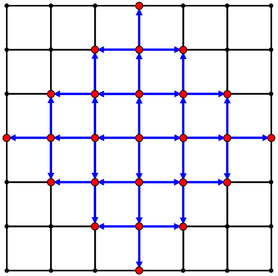
- **Ferromagnetic Ising Models:** [BRZ95, EKPS00]
  Reconstruction impossible on *tree* $\Leftrightarrow$ Free boundary Gibbs state of Ising model on tree is extremal.

How to assess information decay in networks?



Information percolation:

In Graphical Models

$$I(X_a; X_b) \leq \mathrm{perc}(a, b)$$

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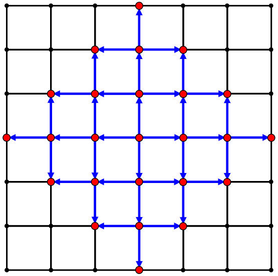

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1. [P.-Wu'16]: "Dissipation of information in channels with input constraints"
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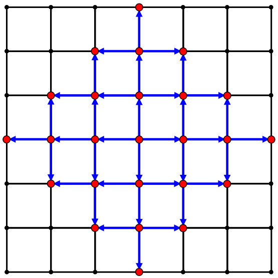

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What is η_{KL}?

Data processing inequality

$$\begin{tikzcd}
P_X \arrow[dr] & & & P_Y \\
& X \arrow[r, "P_{Y|X}"] & Y \arrow[ur] \arrow[dr] & \\
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- In most cases, inequality is strict...

Strong data-processing inequality (SDPI)

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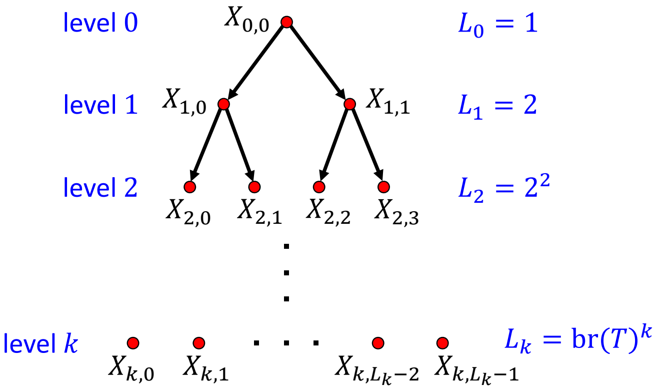
Next: Special case of broadcasting problem

Broadcasting on Trees

- Fix infinite tree T with branching number $\mathrm{br}(T)$.



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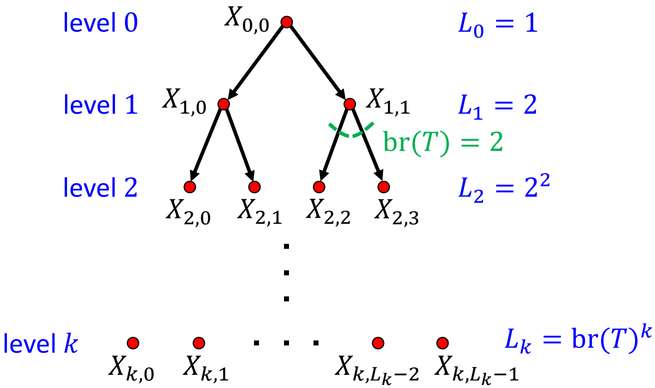

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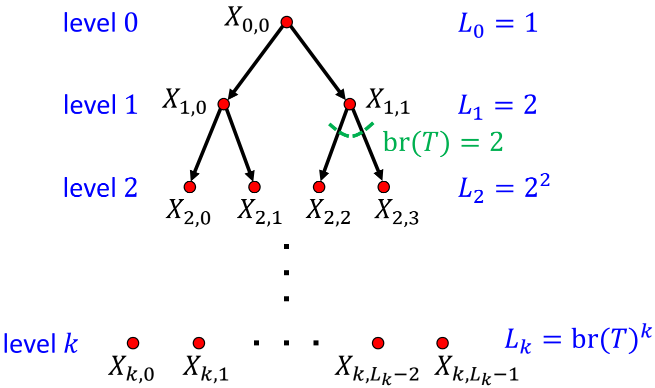

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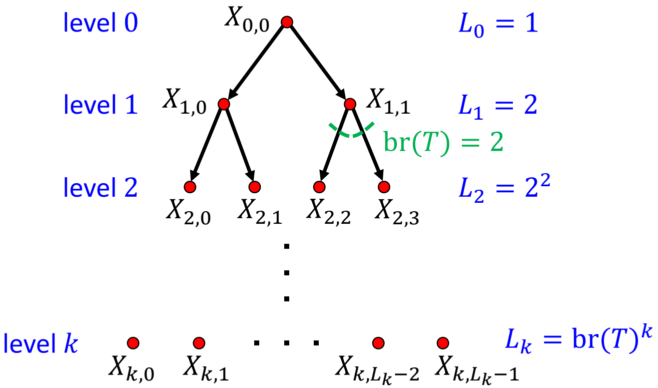

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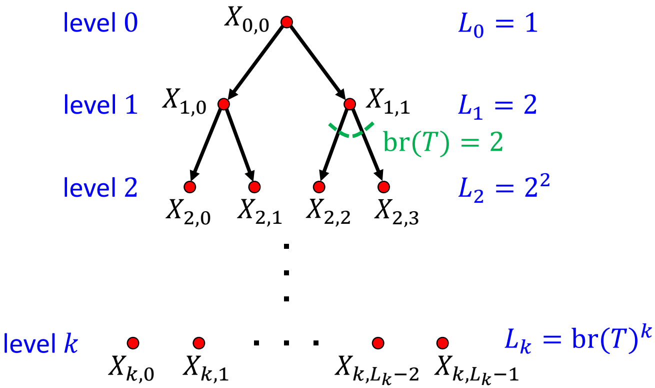
- Edges are independent BSCs with crossover probability $\delta \in \left(0, \frac{1}{2}\right)$.
- Let $P_{\mathsf{ML}}^{(k)} = \mathbb{P}\left(\hat{X}_{\mathsf{ML}}^{k}(X_k) \neq X_{0,0}\right)$, where $X_k = \left(X_{k,0}, \ldots, X_{k,\mathrm{br}(T)^k-1}\right)$.



Broadcasting on Trees: Who cares?

To summarize:

- Root variable $X_{0,0}$ is the information
- It spreads along a tree of $BSC(\delta)$'s.
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This (or similar) question is common:

- Coding: analysis of sparse-graph codes
- CS: Random constraint satisfaction (e.g. k-SAT)
- Stats/ML: Community detection

Direct proof: density evolution

How would a non-IT guy prove it?

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$$S(\mu) = \text{Law of } \ln \frac{\delta e^L + \bar{\delta}}{\bar{\delta} e^L + \delta}, \qquad L \sim \mu$$

$$T(\mu) = \text{Law of } L'(L_1, L_2)\,, \qquad L_1, L_2 \overset{iid}{\sim} \mu$$

and

$$L' = \begin{cases} L_1 + L_2, & \text{w.p. } p(L_1, L_2) + p(-L_1, -L_2) \\ |L_1 - L_2|, & \text{o/w} \end{cases}$$

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Theorem (Phase Transition for Trees [KS66, BRZ95, EKPS00])

- If $\delta < \frac{1}{2} - \frac{1}{2\sqrt{\mathrm{br}(T)}}$, then reconstruction possible: $\lim_{k\to\infty} P_{\mathsf{ML}}^{(k)} < \frac{1}{2}$.
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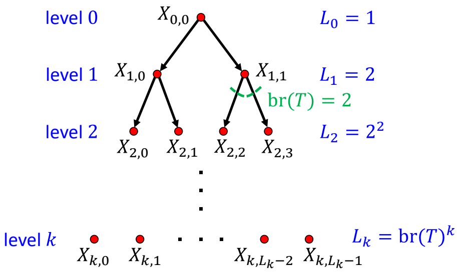

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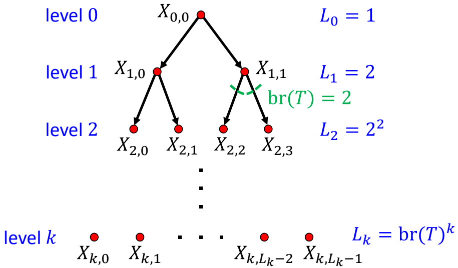

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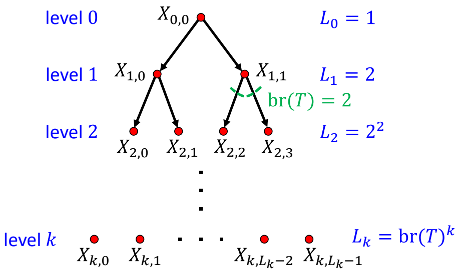

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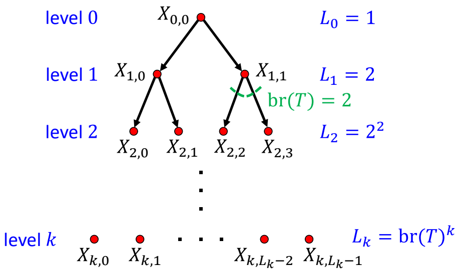

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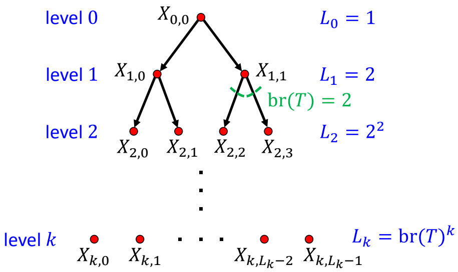
- $\mathrm{br}(T)^k$ paths from X_0 to X_k: $I(X_0;X_k) \le \left(\mathrm{br}(T)(1-2\delta)^2\right)^k$.



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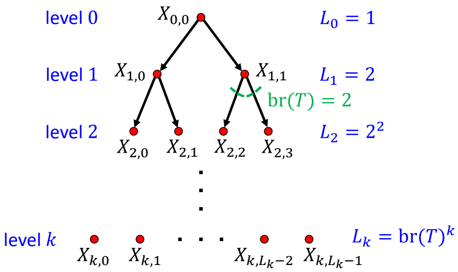
Layers grow by $\mathrm{br}(T)$ and information contracts by $(1-2\delta)^2$. So, whichever effect wins determines reconstruction.



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$$S = \sum_{v \in L_k} f(X_v)$$

  where $f =$ second eigenfunction of the noisy channel.
- For BSC: $f(\sigma) = \sigma, \sigma = \pm 1$.
- Analysis: $\mathbb{E}[S|X_0 = \pm + 1]$ and $\mathrm{Var}[S]$ can be computed easily due to choice of f.
- ... It shows the $X_0 = \pm 1$ can be separated if $\lambda_2^2 \, \mathrm{br}(T) > 1$.

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- Analysis: $\mathbb{E}[S|X_0 = \pm + 1]$ and $\mathrm{Var}[S]$ can be computed easily due to choice of f.
- ... It shows the $X_0 = \pm 1$ can be separated if $\lambda_2^2 \operatorname{br}(T) > 1$.
- In other words, KS corresponds to a suboptimal majority-vote decoder.
- ... and thus results in a suboptimal P_e.
- ... but surprisingly recovers the right threshold for BSC (but not in general, e.g. for Potts with $q = 5$).

Broadcasting on trees: lower bound

- The IT intuition above is awesome.
- Annoyance: lower bound is shown in a very different way!
- Kesten-Stigum bound [KS66]: Let

$$S = \sum_{v \in L_k} f(X_v)$$

  where $f =$ second eigenfunction of the noisy channel.
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- ... and thus results in a suboptimal P_e.
- ... but surprisingly recovers the right threshold for BSC (but not in general, e.g. for Potts with $q = 5$).
- Can we analyze the optimal decoder? (without studying $T \circ S$)
  This is one goal of my talk

Some background first...

- BMS channels
- Channel comparison orders: degraded, more capable, less noisy

BMS channels

Definition

$P_{Y|X} : \{\pm 1\} \to \mathcal{Y}$ called BMS if there is a bijection $h : \mathcal{Y} \to \mathcal{Y}$ s.t.

$$P_{Y|X}(y|x) = P_{Y|X}(h(y)| - x) \qquad \forall x, y$$

- For example, BSC, BEC, BI-AWGN, but also...

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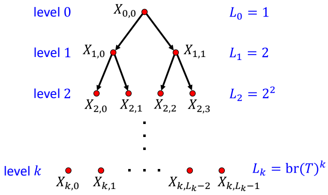

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- Let $X \sim \mathrm{Uniform}\{\pm 1\}$ and define
  1. $P_e = \mathbb{P}[X \neq \hat{X}_{ML}(Y)]$
  2. $C = I(X;Y)$
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- Let $X \sim \mathrm{Uniform}\{\pm 1\}$ and define
  1. $P_e = \mathbb{P}[X \neq \hat{X}_{ML}(Y)] \qquad = \mathbb{E}[|1 - 2\Delta|]$
  2. $C = I(X; Y) \qquad = \log 2 - \mathbb{E}[h(\Delta)]$
  3. $C_{\chi^2} = \chi^2(P_{Y|X=+1} \| P_Y) \qquad = \mathbb{E}[(1 - 2\Delta)^2]$
- Every BMS has a BSC-mixture representation:

$$Y = (\Delta, \mathsf{BSC}_\Delta(X))\,, \quad \Delta \sim P_\Delta \perp\!\!\!\perp X$$

- The evolution operator $T \circ S$ described dist. of $\log \frac{1-\Delta}{\Delta}$.

Less noisy comparison

Definition

We say $P_{Y|X} \le_{ln} P_{Z|X}$ if for every $P_{U,X}$ we have

$$\begin{tikzcd}[row sep=small] & & Y \\ U \arrow[r] & X \arrow[ur] \arrow[dr] & \\ & & Z \end{tikzcd} \qquad \implies I(U;Y) \le I(U;Z)$$

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- The meaning is that $P_{Z|X}$ is a better channel (in the sense above)
- Other partial orders exist: $P_{Y|X} \le_{deg} P_{Z|X}$, $P_{Y|X} \le_{mc} P_{Z|X}$ (degradation, more capable)
- ... we won't need them

Application of channel comparisons

- Consider a system processing X_0 into Y_1, Y_2 as follows (arrows are noisy channels):

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Question: Is the channel $X_0 \to (Z_1, Y_2)$ less noisy than $X_0 \to (Y_1, Y_2)$?

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$$\begin{tikzcd} & & & \color{red}{Z_1} \\ & & X_1 \arrow[ur, dashed] \arrow[r] & Y_1 \\ U \arrow[r] & X_0 \arrow[ur] \arrow[dr] & & \\ & & X_2 \arrow[r] & Y_2 \end{tikzcd}$$

$$\begin{aligned} I(U; Y_1, Y_2) &= I(U; Y_2) + I(U; Y_1 | Y_2) \\ &\le I(U; Y_2) + I(U; Z_1 | Y_2) \qquad \text{by def. of } \le_{ln} \\ &= I(U; Z_1, Y_2) \end{aligned}$$

Comparison method for analyzing networks

$$
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- Alas, it is very hard to prove $\leq_{ln}$ relation... Or is it?

Extremality of BSC and BEC

Theorem (Roozbehani-P.'2019)

1. *Among all BMS channels W with fixed P_e the BSC and BEC are extremal w.r.t. degradation:*

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3. *Among all BMS channels W with fixed C_{χ^2} the BSC and BEC are extremal w.r.t. less noisy:*

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Note: We only care about No.3 here, which is new!

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- In [RP19] we used this to analyze new non-linear sparse-graph codes (LDMCs).
- The proof in fact shows a version of Mrs. Gerbers Lemma:

  divergence $d(p * \delta \| q * \delta)$ is convex in $C_{\chi^2} = (1-2\delta)^2\ \forall p, q \in [0,1]$

  (Usual MGL: $q = 1/2$ and C_{χ^2} replaced with C)

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  (Usual MGL: $q = 1/2$ and C_{χ^2} replaced with C)
- We are ready to get rid of Kesten-Stigum

Theorem (sharp reconstruction lower bound)

Consider b-ary tree. If $b(1-2\delta)^2 > 1$ *then* $\liminf_{d\to\infty} I(X_0; X_{L_d}) > 0$

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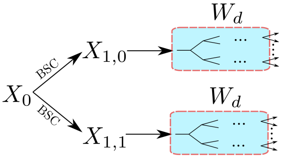

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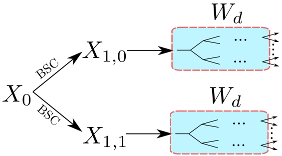

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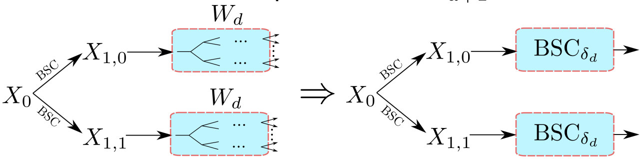

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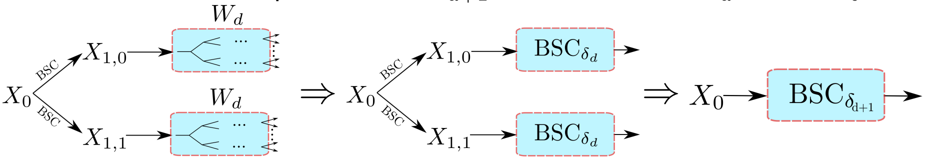

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  where $\delta_{d+1} \triangleq J(\delta_d)$ for some explicit $J(x)$.

- Starting from $\delta_0 = 0$, analysis shows $\delta_\infty < 1/2$. Thus, for all d we have $W_d \geq_{ln} \mathrm{BSC}_{\delta_\infty}$

Define two quantities for $\delta < \delta_{crit}(b) = \frac{1}{2} - \sqrt{\frac{1}{4b}}$:

$$P_e(\delta) \triangleq \lim_{d\to\infty} \mathbb{P}[X_0 \neq \hat{X}_0(X_{L_d})]$$

$$I(\delta) \triangleq \lim_{d\to\infty} I(X_0; X_{L_d})$$

- In physics, behavior of quantities near the phase transition is often universal, e.g. *critical exponents*.
- So we ask: What are α, β, γ?

$$P_e(\delta_{crit} - \tau) = 1/2 - \Theta(\tau^\alpha)$$
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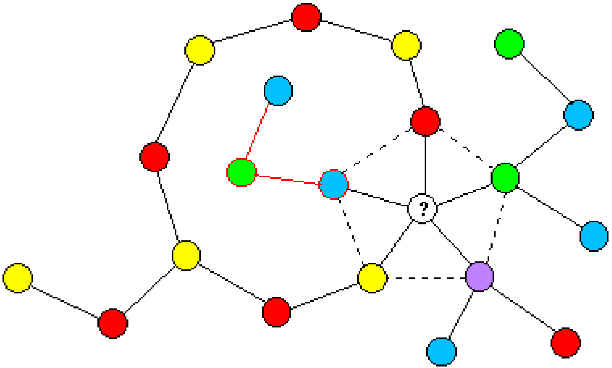
- Previously: $\beta = 1$, $1/2 \le \alpha \le 1$, some loose bounds on γ.
- Our methods (rigorous, except for finite precision arithmetic):

$$\gamma \approx 8\sqrt{2}, \qquad 1/2 \le \alpha \le 0.504$$

Next: Reconstruction on sparse graphs

Reconstructing random colorings



- Consider large sparse graph with randomly colored vertices
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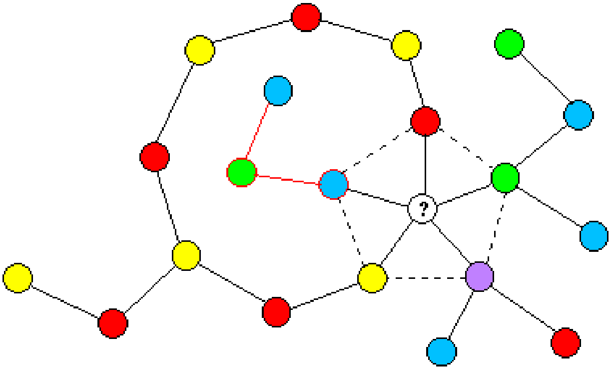

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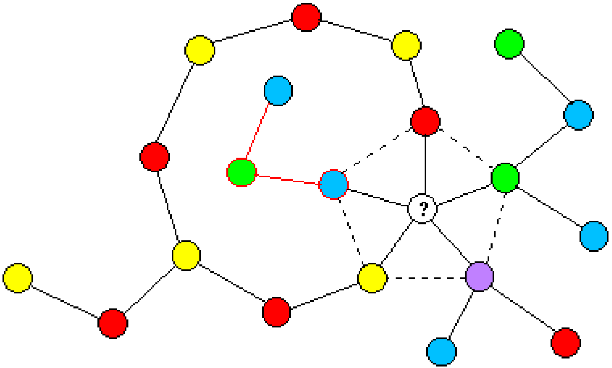

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- Global question: Are there long-range dependencies?
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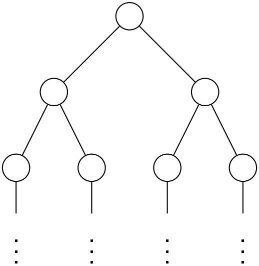
- ... if graph is locally tree-like we get a BoT question!

Reconstructing random colorings: achievability



- Suppose we have k colors and regular graph of degree $d+1$.
- if $d \geq (1+o(1))k \log k$ then w.h.p. each node has among its descendants all colors except its own.

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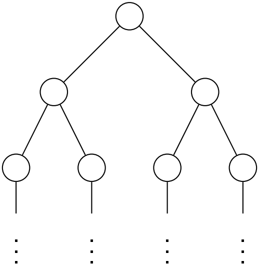

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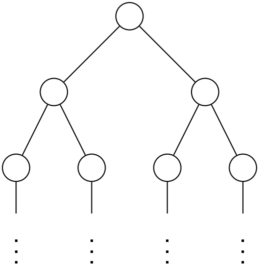

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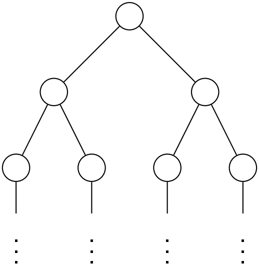

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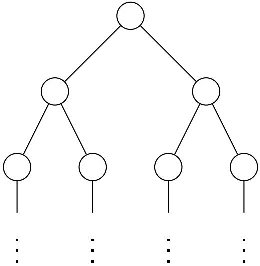
- Yes! Two long papers: [Sly '09], [Bhatnagar-Vera-Vigoda-Weitz'11]

Broadcasting on trees: General edge channel

- Infinite tree $\mathcal{T}$ with marked root ρ.
- Reversible Markov kernel $W : [k] \to [k]$ with invariant distribution q^*.
- Each node has a color in $[k]$, where
  - Root color has distribution q^*.
  - Color of any non-root node is generated from color of its parent by applying W.
- We say the model has non-reconstruction if

$$\lim_{h \to \infty} I(\rho; L^h) = 0,$$

where L^h is the set of nodes on level h.



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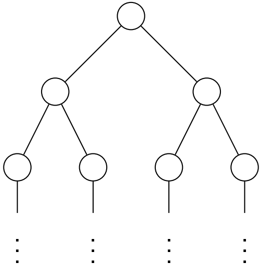
  where L^h is the set of nodes on level h.
- Note: For k-coloring channel $W(y|x) = \frac{1}{k-1}1\{y \neq x\}$



Broadcasting on trees: Result

Theorem (G.-Polyanskiy '19)

Let $\mathrm{br}(\mathcal{T})$ *be the branching number of the tree. Then we have non-reconstruction if*

$$\eta_{\mathsf{KL}}(q^*, W)\,\mathrm{br}(\mathcal{T}) < 1.$$

- If $\mathcal{T}$ is a d-regular tree or a Galton-Watson tree with expected offspring d, then $\mathrm{br}(\mathcal{T}) = d$.

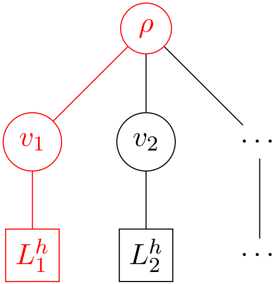
- Apply SDPI to the Markov Chain

$$L_i^h \to v_i \xrightarrow{W} \rho,$$

and get $I(\rho; L_i^h) \le \eta_{\mathsf{KL}}(q^*, W) I(v_i; L_i^h)$.

- Use conditional independence.

$$\begin{aligned} I(\rho; L^h) &\le \sum_i I(\rho; L_i^h) \\ &\le \sum_i \eta_{\mathsf{KL}}(q^*, W) I(v_i; L_i^h) \\ &= d\eta_{\mathsf{KL}}(q^*, W) I(\rho; L^{h-1}). \end{aligned}$$



- Apply induction.

Broadcasting on trees: Comparison

- For the coloring channel $W_{i,j} = \frac{1}{k-1}\mathbb{1}\{i \neq j\}$, we obtain non-reconstruction for

$$d < \frac{\log k}{\log k - \log(k-1)} = (1 - o(1))k \log k.$$

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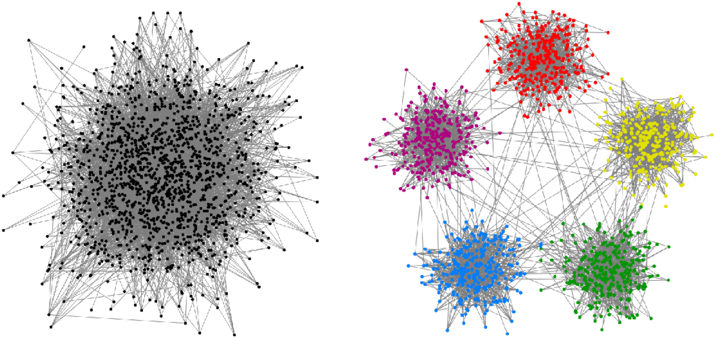
- ... Sharp!
- For Potts channels and binary asymmetric channels, we obtain better numerical values for small d.
- Our results are non-asymptotic in k, d, and work for arbitrary trees.

Application: Community detection



- Unsupervised clustering problem
- See: $0/1$ similarity (i.e. graph)
- Want: Are there any clusters?

Stochastic block model

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- We say weak recovery is possible for parameters (k, a, b) if there exists $\epsilon > 0$ such that, with high probability, given the graph $\mathbb{G}$, we can construct a partition of the vertex set that is correct for at least ϵn vertices.

Stochastic block model: Result

Theorem (G.-Polyanskiy '19)

Let $d = \frac{a+(k-1)b}{k}$, $\lambda = \frac{a-b}{a+(k-1)b}$. *Weak recovery is impossible if*

$$d\eta_{\mathsf{KL}}(\mathrm{PC}_\lambda, q^*) < 1,$$

where PC_λ *is the Potts channel defined by*

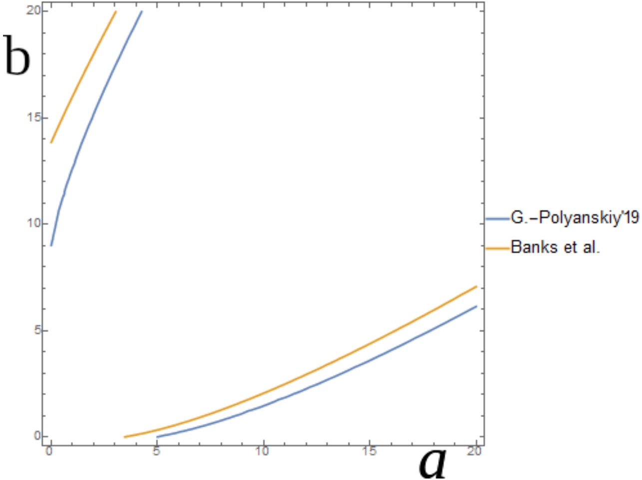
$$\mathrm{PC}_\lambda(x,y) = \frac{1-\lambda}{k}\mathbb{1}\{x \neq y\} + (\frac{1}{k} + \frac{k-1}{k}\lambda)\mathbb{1}\{x = y\}.$$

and q^* *is the uniform distribution.*

Proof.

Reduction from broadcasting on trees. ∎

Stochastic block model: Comparison



- For $k \geq 3$, $a > b$, we improve the state-of-the art [Banks et al. '16].
- Note: for $a < b$, exact threshold is known [Coja-Oghlan et al. '19.]

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Conclusion

- Machine learning: exciting new **local to global** problems
- ... sometimes called combinatorial statistics
- Obvious connections with statistical physics
- Information theory: excellent tools for these problems
- Previously: only on the negative (impossibility) side and "easy" problems
- New: channel comparison, SDPI, info-percolation
- ... positive results, sharp threshoulds, hard models

Thank You!

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