An Improved Lower Bound on Rate for Variable-length Codes with Active Feedback

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Overview

- **Main result**: Improved lower bound on maximum rate of variable-length feedback codes at short blocklengths
- Previous lower bound [Polyanskiy, Poor and Verdú, 2011]: stop-feedback codes, left large gap to upper bound
- New approach: "active" feedback to confirm receiver's estimate
- Numerical results provided for BSC

VLF Codes

An (l, M, ϵ) **variable-length feedback (VLF) code** consists of [Polyanskiy, Poor and Verdú, 2011]:

- Message $W \in \{1, 2, ..., M\}$
- Average blocklength l: $\mathrm{E}[\tau] \leq l$
- τ is a stopping time of the filtration $\sigma\{U, Y_1, Y_2, ...\}$
- U is common randomness revealed to both Tx and Rx
- Encoder outputs $X_n = f_n(U, W, Y_1, Y_2, ..., Y_{n-1})$
- Memoryless channel $\mathrm{P}(Y_i \mid X_1, ..., X_i) = \mathrm{P}(Y_i \mid X_i)$
- Decoder's estimates $g_n(U, Y_1, ..., Y_n)$
- Decoder's final decision $\hat{W} = g_\tau(U, Y_1, ..., Y_\tau)$
- Average probability of error ϵ s.t. $\mathrm{P}[\hat{W} \neq W] \leq \epsilon$
- Code rate is $(\log M) / l$

Stop-feedback

Stop-feedback VLF code:

- Tx ignores feedback except to learn when Rx stops transmission (decodes)
- Encoder outputs $X_n = f_n(U, W)$
- Also called **decision feedback** (ACK/NACK from Rx)

Finite-blocklength regime:

- Feedback improves the maximum rate at short blocklengths compared to no-feedback case. (Fig. 1)
- Large gap between lower (achievability) and upper (converse) bounds on rate.
- Best achievability result for DMCs based on stop-feedback codes – Doesn't consider what receiver knows!

[PPV'11]: Y. Polyanskiy, H. V. Poor, and S. Verdú, "Feedback in the non-asymptotic regime," IEEE Trans. Inf. Theory, 2011.

Stop-feedback Bound

Theorem: (Stop-feedback) Achievability [PPV'11, Thm. 3]

For a scalar $\gamma > 0$, there exists an (l, M, ϵ) VLF code satisfying

$$l \leq \mathrm{E}[\tau]$$
$$\epsilon \leq (M-1)\,\mathrm{P}[\bar{\tau} \leq \tau],$$
$$\tau = \inf\{n \geq 0: i(X^n;Y^n) \geq \gamma\},$$
$$\bar{\tau} = \inf\{n \geq 0: i(\bar{X}^n;Y^n) \geq \gamma\}.$$

- $i(X^n;Y^n)$ is the information density between codeword X^n and channel output Y^n.
- $i(\bar{X}^n;Y^n)$ is the information density between identically-distributed codeword $\bar{X}^n$ and channel output Y^n.
- *Proof*: Random coding argument.

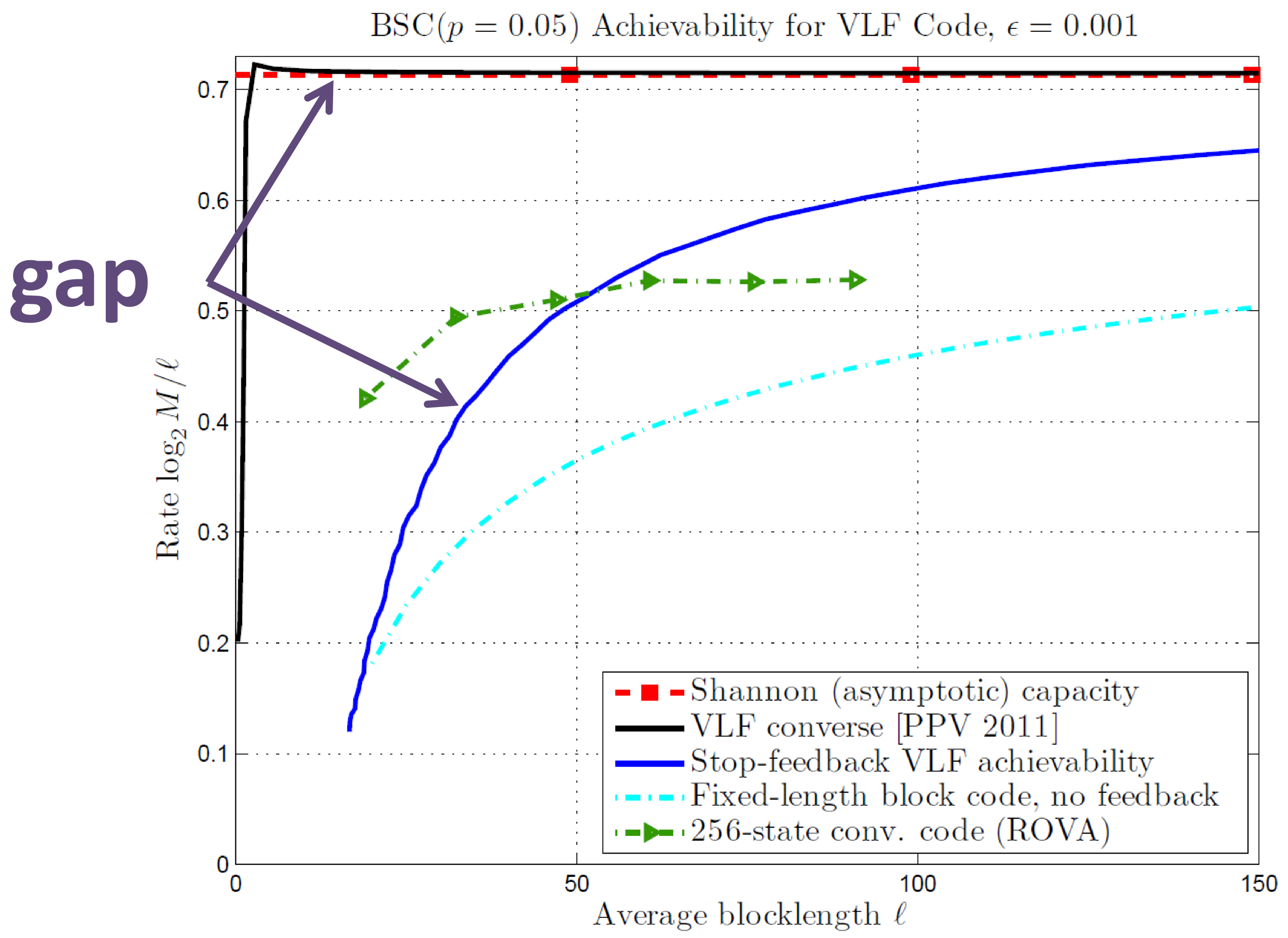

Fig. 1: Gap between upper and lower bounds on max. rate at short blocklengths. Feedback provides improvement vs. no-feedback. ROVA = Reliability Output Viterbi Algorithm [ISIT '13].

"Active" Feedback

- Transmitter uses feedback to refine receiver's tentative estimate.
- In general, $f_n(U, W, y^{n-1}) \neq f_n(U, W, \tilde{y}^{n-1})$, when $y^{n-1} \neq \tilde{y}^{n-1}$
- Channel coding is a specific case of **active sequential hypothesis testing** [Naghshvar and Javidi, 2012].
- Benefit of active feedback called **adaptivity gain**.
- Active feedback also called **information feedback**.

[ISIT '13]: A. R. Williamson, T.-Y. Chen, and R. D. Wesel, "Reliability-based error detection for feedback communication with low latency," IEEE Int. Symp. Inf. Theory, 2013.
[Naghshvar and Javidi, 2012]: M. Naghshvar and T. Javidi, "Sequentiality and adaptivity gains in active hypothesis testing," arXiv, 2012.

Improved Lower Bound

Proposed scheme:

- Decoder feeds back estimate X^n once $i(X^n;Y^n) \geq \gamma$ for some X^n
- Tx uses N forward symbols to confirm (ACK) or deny (NACK) estimate
- Start over if Rx decodes NACK, stop when Rx decodes ACK
- $\mathrm{P}[n \to a] = \mathrm{P}\{\text{NACK decoded as ACK}\}$
- $\mathrm{P}[a \to n] = \mathrm{P}\{\text{ACK decoded as NACK}\}$
- $\mathrm{P(NACK)} = \mathrm{P}\{\text{Rx decodes NACK}\}$

$$= \mathrm{P}[n \to n]\mathrm{P}\{\text{Rx est. wrong}\} + \mathrm{P}[a \to n]\,\mathrm{P}\{\text{Rx est. correct}\}$$
$$\leq \mathrm{P}[n \to n](M\text{-}1)\,\mathrm{P}[\bar{\tau} \leq \tau] + \mathrm{P}[a \to n]$$

Theorem: Improved Achievability for Active Feedback

For a scalar $\gamma > 0$ and integer $N > 0$, there exists an (l, M, ϵ) VLF code satisfying

$$l \leq \frac{\mathrm{E}[\tau] + N}{1 - \mathrm{P(NACK)}}$$
$$\epsilon \leq \frac{(M-1)\,\mathrm{P}[\bar{\tau} \leq \tau]\,\mathrm{P}[n \to a]}{1 - \mathrm{P(NACK)}}.$$

- *Proof*: Similar to stop-feedback proof.
- Numerical evaluation (Fig. 2) requires optimization over γ, N, and threshold N_t (threshold for skewed hypothesis test of confirmation block at Rx), for fixed M and ϵ.

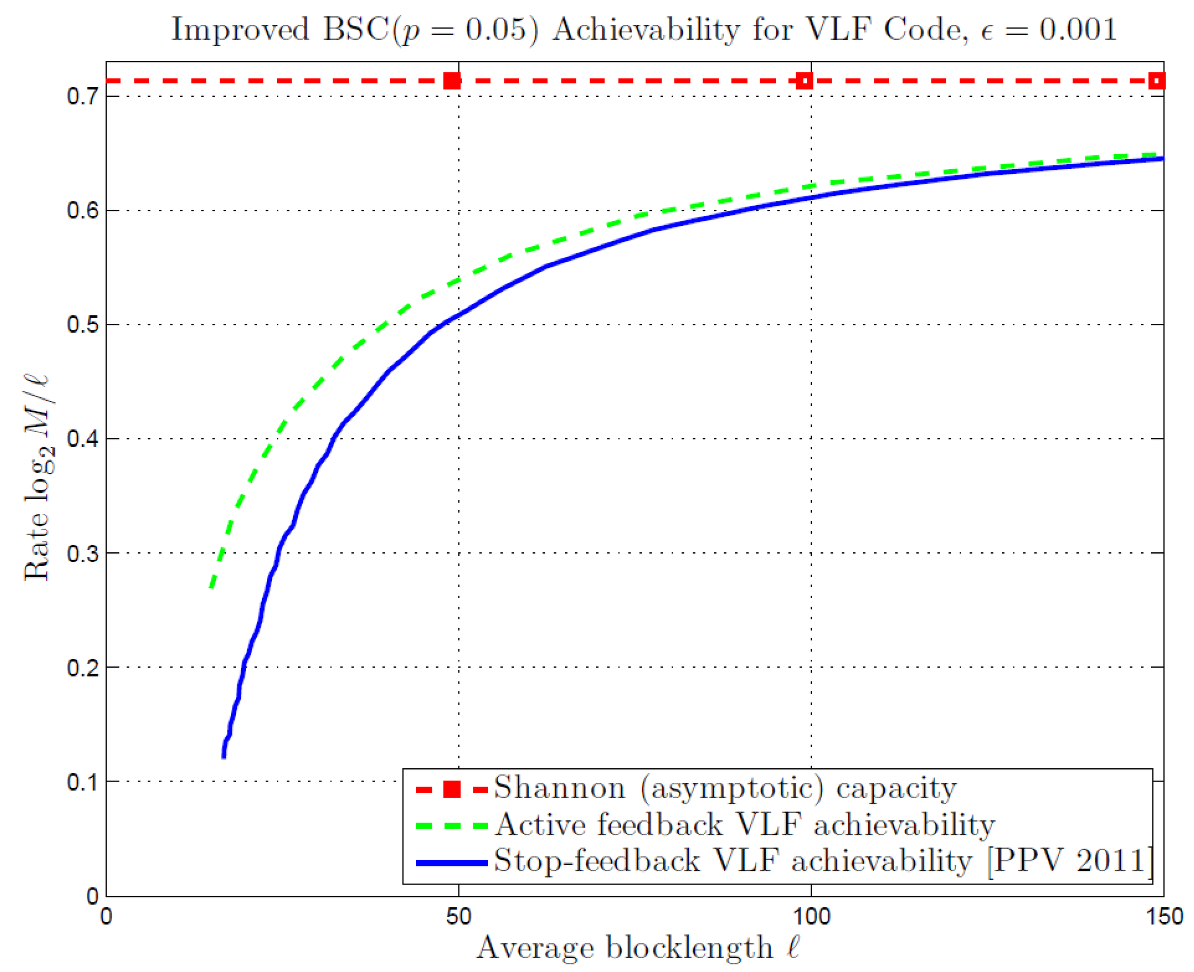

Fig. 2: Numerical evaluation of new "active" feedback lower bound.

Discussion

- Can do better by refining Rx estimate sequentially, not just at τ
- Starting over after NACK is costly in terms of latency
- Still need to find "good" codes
- There may be encoder complexity challenges