**Communication system**

- Vector $\mathbf{x} \in [0,1]^k$ drawn from a discrete-time continuous alphabet source transmitted over an $n$-dimensional Gaussian channel

$$x \in \mathbb{R}^k, \quad y \in \mathbb{R}^k, \quad z \sim \mathcal{N}(0, \sigma^2 I_n)$$

1. Bandwidth expansion ratio $n/k$
2. Under constraint $\mathbb{E} \left[ \left\| \mathbf{s}(\mathbf{x}) \right\|^2 \right] \leq P$, minimize MSE
   $$\text{MSE} = \frac{1}{k} \mathbb{E} \left[ \left\| \mathbf{x} - \hat{\mathbf{x}} \right\|^2 \right].$$

**Information-Theoretical Limits**

- Rate-Distortion Theory + Separation Principle:
  $$D \geq \frac{1}{2 \pi e (1 + \text{SNR})^{n/k}}$$
- Achievable with arbitrarily long (digital) block codes and infinite delay.
- Question: How to design explicit and efficient analog mappings
  $$\mathbf{s} : [0,1]^k \rightarrow \mathbb{R}^n$$
  with asymptotically optimal behavior
  $$\text{MSE} = \Theta(\text{SNR}^{-n/k})$$
  - If $\mathbf{s}(\mathbf{x})$ is linear, then $\text{MSE} = \Theta(\text{SNR}^{-1})$. Thus we must consider non-linear functions.

**Cramér-Rao Bound and Low-Noise Approximation**

The Cramér-Rao bound on the MSE for this model can be evaluated as

$$\frac{1}{k} \mathbb{E} \left[ \left\| \mathbf{x} - \hat{\mathbf{x}} \right\|^2 \right] \geq \frac{\sigma^2}{k} \int_{[0,1]^k} \text{tr}(J(\mathbf{x})^t J(\mathbf{x}))^{-1} d\mathbf{x},$$

where $J(\mathbf{x})$ is the Jacobian of $\mathbf{s}(\mathbf{x})$. In fact, if the noise is small (smaller than the distance between two “segments” of the curve), the MSE is well approximated by the CR-bound.

**Threshold Effect**

Design criteria:

- Maximize distance between “segments” of the locus $\mathbf{s}(0^k,1^k)$.
- “Stretch” the locus as much as possible/equally in each direction.

**The mod-1 map**

For $A \in \mathbb{Z}^n \times k$, we consider the piecewise linear map

$$s_i(\mathbf{x}) = (A(\mathbf{x}))_i := A\mathbf{x} \mod 1 = A\mathbf{x} - \lfloor A\mathbf{x} \rfloor.$$

The map is injective if $A$ is a primitive set of vectors in $\mathbb{Z}^n$ (i.e., can be completed to a basis). Image consists of parallel “planes” inside the box $[-1/2,1/2]^n$.

Distance between two segments:

$$\delta = \min_{n \in \mathbb{Z}^p} \min \left\| A\mathbf{x} - n \right\|$$

= the norm of the shortest vector in the lattice obtained by the projection of $\mathbb{Z}^p$ onto $A^\perp$. Tradeoff between minimum distance/determinant:

$$\rho = \frac{\alpha \delta^2}{2} = \frac{2 \sqrt{\det(A^\perp A)^{1/(n-k)}}}{\sqrt{n \det(A^\perp A)^{1/(n-k)}}}.$$

**Analysis of the map**

When there are no large errors:

$$\text{MSE} \approx \frac{\sigma^2}{12 k^P} \frac{n}{\text{det}(A^\perp A)^{-1}},$$

but to meet the small error conditions we need $\rho$ to be large $\Rightarrow$ $\text{det}(A^\perp A)$ small. To achieve optimal exponent we need a family of matrices with:

1. (Injectivity) The columns of $A$ are primitive.
2. (Minimum distance) The density of the projections of $\mathbb{Z}^n$ onto $A^\perp$ is bounded away from zero.
3. (MSE Exponent) $\text{tr}(A^\perp A)^{-1} = O(\text{det}(A^\perp A)^{-1/k})$.

(3.) is trivially satisfied if $A$ is orthogonal. For some parameters ($k = n - 2, n - 1$) constructions are possible. However, orthogonality + primitivity + good projections are hard to ensure simultaneously.

**Ex: $n - 1$ to $n$**

Consider the matrix:

$$A_w = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & w
\end{bmatrix}$$

Condition (1) and (2) are straightforward. Condition (3):

$$\text{det} A_w^w A_w = \Theta(\text{w}^{2n-2})$$

and $[(A_w^w A_w)^{-1}]_{ij} = O(1/w^2)$. The associated mod-1 maps will have optimal exponent.

**An Alternative Mapping: Modifying the support**

By a matrix factorization, we can find $Q$ and $R$, where $\det R = 1$ and columns of $Q$ orthogonal, such that $A = QR$. Then the mapping:

$$s_Q : S \rightarrow \mathbb{R}^n$$

where $S = R[0,1]^k$ yields an asymptotically optimal family (provided that $A$ is chosen according to good projections). However the source is now $S \neq [0,1]^k$ a parallelogram. If $R^{-1}$ is applied to go back to $[0,1]^k$, it is possible that small errors will be magnified. To go back to the support $[0,1]^k$ we need an application that acts like an isometry.

**Dissections of polyhedra**

Idea: use $s_Q$ and a bijection between the cube $[0,1]^k$ and $S$ provided by a dissection to come back to the original support.

- **Dissect** $[0,1]^k$ and $S$ into $m$ non-overlapping polyhedra $T_1, T_2, \ldots, T_m$ and $S_1, S_2, \ldots, S_m$ so that $[0,1]^k = \bigcup_{i=1}^m T_i$, $S = \bigcup_{i=1}^m S_i$ and $T_i = \phi(S_i)$, where $\phi$ is an isometry. Define the map $s(\mathbf{x}) = s_Q(\phi(S_i))$ if $\mathbf{x} \in T_i$. Discontinuities can cause large errors. Solution: shrinking factor.

**Proposition:** For $k = 2$, there is a family of matrices and a proper choice of the shrinking factor such that $\text{MSE} = \Theta(\text{SNR}^{-n/2})$.

(very short) sketch of the proof: Choose a sequence of projections similar to $[2]$ that exhibits optimal behavior after dissecting before reassembling. If the shrinking factor is chosen properly, the degradation caused by the dissection technique is exponentially small, keeping the right behavior.

**References**


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