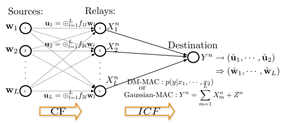
The capacity region of three user Gaussian inverse-compute-and-forward channels

Yanying Chen, Yiwei Song, Natasha Devroye
University of Illinois at Chicago, Chicago IL 60607, USA

Inverse-Compute-and-Forward (ICF) problem **(Motivation / Background)**

• Compute-and-Forward (CF): enables the decoding of linear combinations of messages at relays over Gaussian channels.
• Inverse Compute-and-Forward (ICF): to extract or decode individual messages over a MAC from relays which possess linear combinations of messages, previously obtained through CF.



• **ICF problem:** What are the rate constraints on original message $\mathbf{w}$'s such that the destination can recover all individual messages via a MAC from linear equations available at relays ?

•Note the convex hull of the intersection of CF-region and ICF-region serves as one achievable rate region for the whole network

Intuition

Intuitively, there are two ways of improving upon the MAC region:
1) the equations are correlated, so if one equation is correct/wrong at the destination this will impact the number of possible choices of values the others may take, thereby reducing the rates on the left-hand side (LHS) of capacity expressions;
2) Common messages may be extracted from the equations which may be coherently sent over the Gaussian MAC, intuitively lifting the right-hand side (RHS) of the capacity region expressions.

Main Contributions **Theorem [3, 4]**

• DM-MAC [Theorem 3]
one achievable scheme with codebook structure

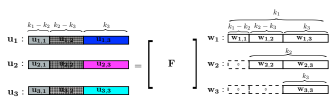
$$p(q)p(x_1|q)p(x_2|q)p(x_3|q)$$

• Gaussian-MAC [Theorem 4]
The following coding scheme is shown to be optimal

$$p^{\mathcal{G}}(q)p^{\mathcal{G}}(x_1|q)p^{\mathcal{G}}(x_2|q)p^{\mathcal{G}}(x_3|q)$$

✓ **The converse is our main technical contribution.** The key idea is to exploit the linearity and second moment constraints of the AWGN channel and note that independent Gaussian codewords are sufficient for both the pairwise independent and independent components

Equation Structure at relays **Lemma [1,2]**



Three user ICF message / equation structure. $\mathbf{w}_l \in \mathbb{F}_p^l, f_{ml} \in \mathbb{F}_p$.

• Equation sections $\mathbf{u}_{1,1}, \mathbf{u}_{2,1}, \mathbf{u}_{3,1}$ (or $\mathbf{U}_{,1}$) are fully correlated, and may be used to reconstruct $\mathbf{w}_{1,1}$, a common message known to all relays. (grey color)
• $\mathbf{u}_{1,2}, \mathbf{u}_{2,2}, \mathbf{u}_{3,2}$ are pairwise independent. Specifically, any two of these are pairwise independent and the third is a deterministic function of the other two. The three are not mutually independent. (shading)
• $\mathbf{u}_{1,3}, \mathbf{u}_{2,3}, \mathbf{u}_{3,3}$ are mutually independent. (different solid colors)

Assumption: The coefficient matrix $\mathbf{F}$ is nonsingular for the feasibility and we furthermore assume here that all of its square sub-matrices are full rank.

One achievable scheme **Theorem [3]**

Theorem 3 (DM 3-user ICF achievability): The messages $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ at rates $(R_1 \geq R_2 \geq R_3)$ may be recovered from $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ (where $\mathbf{U} = \mathbf{F} \cdot \mathbf{W}$) sent over a MAC, assuming $\mathbf{F}$ and all of its square sub-matrices are non-singular, if the rates lie in

$$\mathcal{R}_{IN} := \bigcup_{p(q)p(x_1|q)p(x_2|q)p(x_3|q)} \mathcal{R} \tag{2}$$

for $|Q| \leq \min\{|\mathcal{X}_1| \cdot |\mathcal{X}_2| \cdot |\mathcal{X}_3|, |\mathcal{Y}|\}$, where $\mathcal{R}$ is defined as the set of (R_1, R_2, R_3) with $(R_1 \geq R_2 \geq R_3)$ and

$$R_1 + R_2 + R_3 \leq I(X_1, X_2, X_3; Y) \tag{3a}$$
$$2R_2 + R_3 \leq I(X_1, X_2, X_3; Y|Q) \tag{3b}$$
$$R_2 + R_3 \leq I(X_1, X_2; Y|X_3, Q) \tag{3c}$$
$$R_2 + R_3 \leq I(X_1, X_3; Y|X_2, Q) \tag{3d}$$
$$R_2 + R_3 \leq I(X_2, X_3; Y|X_1, Q) \tag{3e}$$
$$R_3 \leq I(X_1; Y|X_2, X_3, Q) \tag{3f}$$
$$R_3 \leq I(X_2; Y|X_1, X_3, Q) \tag{3g}$$
$$R_3 \leq I(X_3; Y|X_1, X_2, Q) \tag{3h}$$

Codebooks: $Q^n(\mathbf{U}_{,1}) \sim \prod_{t=1}^{n} p(q_t)$, and $X_m^n(\mathbf{u}_{m,2}, \mathbf{u}_{m,3}|\mathbf{u}_{m,1}) \sim \prod_{t=1}^{n} p(x_{mt}|q^n(\mathbf{u}_{m,1}))$, $m = 1, 2, 3$.

Joint typicality decoding :
$(Q^n(\mathbf{U}_{,1}), X_1^n(\mathbf{u}_1), X_2^n(\mathbf{u}_2), X_3^n(\mathbf{u}_3), Y^n)$

The Capacity region **Theorem [4]**

Theorem 4 (3-user Gaussian ICF capacity): The capacity region for recovering messages $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ at rates $R_1 \geq R_2 \geq R_3$ from the equations $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ (where $\mathbf{U} = \mathbf{F} \cdot \mathbf{W}$) sent over an AWGN-MAC in (1), assuming $\mathbf{F}$ and all of its square sub-matrices are non-singular, is

$$\bigcup_{0 \leq b_1, b_2, b_3 \leq 1} \mathcal{R}^{\mathcal{G}}(b_1, b_2, b_3) \tag{4}$$

where

$$\begin{aligned} \mathcal{R}^{\mathcal{G}}(b_1, b_2, b_3) = \Big\{ & (R_1, R_2, R_3) : R_1 \geq R_2 \geq R_3 \\ & R_1 + R_2 + R_3 \leq C(P_1 + P_2 + P_3 + 2\sqrt{b_1 b_2 P_1 P_2} + \\ & \qquad 2\sqrt{b_1 b_3 P_1 P_3} + 2\sqrt{b_2 b_3 P_2 P_3}) \\ & 2R_2 + R_3 \leq C((1-b_1)P_1 + (1-b_2)P_2 + (1-b_3)P_3) \\ & R_2 + R_3 \leq \min_{i \neq j \in \{1,2,3\}} C((1-b_i)P_i + (1-b_j)P_j) \\ & R_3 \leq \min_{i \in \{1,2,3\}} C((1-b_i)P_i) \Big\}. \end{aligned} \tag{5}$$

• The achievability follows from a restriction of Gaussian input distributions in Theorem 3.

Choose: $Q = T_0$,
$X_i = \sqrt{b_i}\sqrt{P_i}T_0 + \sqrt{1-b_i}\sqrt{P_i}T_i$, $i = 1, 2, 3$
where $[T_0, T_1, T_2, T_3] \sim \mathcal{N}(\mathbf{0}_{4\times1}, \mathbf{I}_{4\times4})$
and $b_1, b_2, b_3 \in [0, 1]$.

• The converse.