Hyperbolic lattices obtained from arithmetic Fuchsian groups via hyperbolic tesselations

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Introduction

The signal design problem in the hyperbolic plane seeks to determine lattices (orders of quaternion algebras) from which signal constellations are constructed. More specifically, a signal constellation is a quotient of an order by one of its ideals. The study of maximal orders has its motivation based on the importance that geometrically uniform codes and space-time block codes have in the design of new efficient digital communication systems. The objective of this work is to relate Fuchsian groups to maximal quaternion orders. This enables us to construct lattices in the hyperbolic plane which in turn are used to design signal constellations. The reason for considering maximal orders is that they naturally produce a complete labeling of the points of the constellations.

Quaternion Algebra and Maximal Order

Let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a *quaternion algebra* over a field $\mathbb{K}$ with basis $\{1, i, j, k\}$ satisfying

$$i^2 = \alpha,\ j^2 = \beta \text{ and } k = ij = -ji, \tag{1}$$

where $\alpha, \beta \in \mathbb{K}/\{0\}$. Consider $\varphi : \mathcal{A} \longrightarrow M(2, \mathbb{K}(\sqrt{\alpha}))$ where

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(i) = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix}, \varphi(j) = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \varphi(k) = \begin{pmatrix} 0 & \sqrt{\alpha} \\ -\beta\sqrt{\alpha} & 0 \end{pmatrix}.$$

So φ is an isomorphism of $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ in the subalgebra $M(2, \mathbb{K}(\sqrt{\alpha}))$. Each element of $\mathcal{A}$ is identified with

$$x \longmapsto \varphi(x) = \begin{pmatrix} x_0 + x_1\sqrt{\alpha} & x_2 + x_3\sqrt{\alpha} \\ \beta(x_2 - x_3\sqrt{\alpha}) & x_0 - x_1\sqrt{\alpha} \end{pmatrix}. \tag{2}$$

There is a natural involution in $\mathcal{A}$, which in a basis satisfying (1) is given by $x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \bar{x} = x_0 - x_1 i - x_2 j - x_3 k \in \mathcal{A}$. We define the *reduced trace* and the *reduced norm* on $x \in \mathcal{A}$ by $Trd(x) = x + \bar{x}$ e $Nrd(x) = x \cdot \bar{x}$, respectively. Also, the *discriminant* $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined by the product of the prime ideals at which $\mathcal{A}$ is ramified.
Let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a quaternion algebra and R be a ring with field of fractions $\mathbb{K}$. An *order* $\mathcal{O} = (\alpha, \beta)_R$ in $\mathcal{A}$ is a subring of $\mathcal{A}$ containing 1, equivalently, it is a finitely generated R-module such that $\mathcal{A} = \mathbb{K}\mathcal{O}$.

Proposition [1]: *If $y \in \mathcal{O}$, then y is integral over R, that is, $Tr(y), Nrd(y) \in R$.*

Proposition [3]: *Let $\mathcal{O}$ have a free R-basis $\{y_1, y_2, y_3, y_4\}$. Then the reduced discriminant $\mathcal{D}(\mathcal{O})$ is the square root of the ideal principal $R \cdot det(Tr(y_i\overline{y_j}))$.*

An order $\mathcal{M} \supseteq \mathcal{O}$ is *maximal* if it is not properly contained in any other order. If $\mathcal{M} \supseteq \mathcal{O}$ is a maximal order in $\mathcal{A}$ then the reduced discriminant satisfies $\mathcal{D}(\mathcal{M}) = \mathcal{D}(\mathcal{A})$.

Hyperbolic Geometry and Fuchsian Groups

We will consider two Euclidean models for the hyperbolic plane: the *Upper-half plane* $\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ and the *Poincaré disc* $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$.
Let $PSL(2, \mathbb{R})$ be the set of all Möbius transformations, $T : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$, given by $T_A(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. A Fuchsian group Γ is a discrete subgroup of $PSL(2, \mathbb{R})$. If $f : \mathbb{H}^2 \longrightarrow \mathbb{D}^2$ is given by

$$f(z) = \frac{zi + 1}{z + i}, \tag{3}$$

then $\Gamma = f^{-1}\Gamma_p f$ is a subgroup of $PSL(2, \mathbb{R})$, where $T_p : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ and $T_p \in \Gamma_p < PSL(2, \mathbb{C})$ is such that $T_p(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$. Furthermore, $\Gamma \simeq \Gamma_p$.

Theorem [2]: *The group $\Gamma[\mathcal{A}, \mathcal{O}]$ associated to quaternion algebra $\mathcal{A}$ and quaternion order $\mathcal{O}$ is isomorphic to $PSL(2, \mathbb{R})$. Therefore, $\Gamma[\mathcal{A}, \mathcal{O}]$ is a Fuchsian group called arithmetic Fuchsian group and the order $\mathcal{O}$ is called hyperbolic lattice.*

The Fuchsian Group Γ_{4g}

Let $\{4g, 4g\}$ be a self-dual tessellation with $g \geq 2$ in the hyperbolic plane and P_{4g} the associated regular hyperbolic polygon.

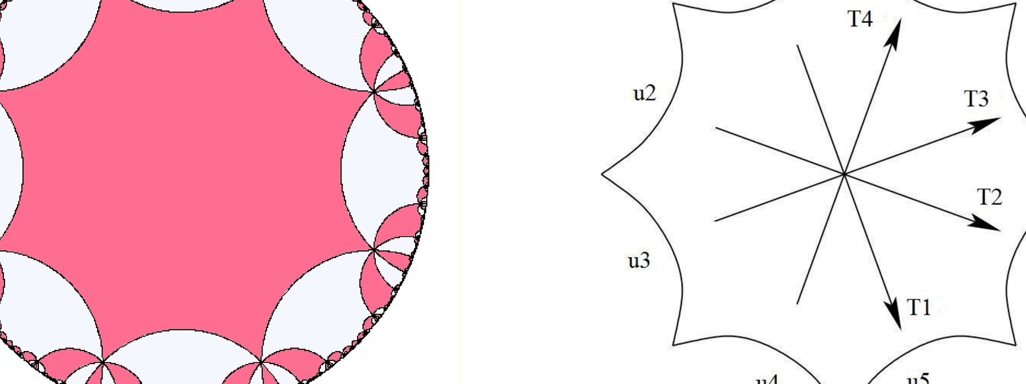

Figure : Hyperbolic tesselation $\{8, 8\}$ and P_8-diametrically opposed edge-pairings

Let the edges of P_{4g} are ordered as follows $u_1, \ldots, u_{4g}$, such that $T_i(u_i) = u_{i+2g}$, $i = 1, \ldots, 2g$. If we have the transformation $T_1 \in \Gamma_{4g}$ and so the corresponding matrix A_1, the remaining generators are obtained by conjugations of the form

$$A_i = C^{i-1}A_1C^{-(i-1)},\ i = 2, \ldots, 2g, \tag{4}$$

where $C = \begin{pmatrix} e^{\frac{i\pi}{4g}} & 0 \\ 0 & e^{-\frac{i\pi}{4g}} \end{pmatrix}$ is the matrix corresponding to the elliptic transformation with order $4g$.
The next result give us the form of the matrix A_1:

Theorem [4]: *Let P_p be a hyperbolic regular polygon with p edges and Γ_p the Fuchsian group associated with the tessellation $\{p, q\}$. If $T_1 \in \Gamma_p$ is such that $T_1(u_1) = u_{1+\frac{p}{2}}$ then the matrix A_1 associated with the transformation T_1 is given by*

$$A_1 = \begin{pmatrix} \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} & \frac{\sqrt{2\cos\frac{\pi}{p} + 2\cos\frac{\pi}{q}} \cdot e^{i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} \\ \frac{\sqrt{2\cos\frac{\pi}{p} + 2\cos\frac{\pi}{q}} \cdot e^{-i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} & \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} \end{pmatrix}. \tag{5}$$

The process of identifying Fuchsian groups derived from a quaternion algebra over a totally real algebraic number field are given by the following results:

Theorem: *For each $g = 2^n, 3 \cdot 2^n, 5 \cdot 2^n$ and $3 \cdot 5 \cdot 2^n$, where $n \in \mathbb{N}$, the elements of a Fuchsian group Γ_{4g} are identified, via isomorphism, with the elements of an order $\mathcal{O} = (\theta, -1)_{\mathbb{Z}[\theta]}$, where*

$$\theta = \begin{cases} \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}} & , \text{ for } g = 2^n; \\ \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \sqrt{3}}}} & , \text{ for } g = 3 \cdot 2^n; \\ \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \frac{\sqrt{10+2\sqrt{5}}}{2}}}} & , \text{ for } g = 5 \cdot 2^n; \\ \sqrt{2 + \sqrt{2 + \ldots + \frac{\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}}{2}}} & , \text{ for } g = 3 \cdot 5 \cdot 2^n. \end{cases} \tag{6}$$

Theorem: *For each $g = 2^n, 3 \cdot 2^n, 5 \cdot 2^n$ and $3 \cdot 5 \cdot 2^n$, with $n \in \mathbb{N}$, the Fuchsian group Γ_{4g}, associated with the hyperbolic polygon P_{4g}, is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{K}}$, over the number field $\mathbb{K} = \mathbb{Q}(\theta)$, where $[\mathbb{K} : \mathbb{Q}] = 2^n, 2^{n+1}, 2^{n+2}$ and 2^{n+3}, respectively, and θ is as in* **(6)**.

Construtiom of the arithmetic Fuchsian group Γ_8 using a maximal order

Now we will show that the arithmetic Fuchsian group Γ_8 can be construct using a maximal order $\mathcal{M}$. In this way, we can produce a complete labeling of the points of the associated constellation.
Let $\Gamma_8 = \langle T_1, T_2, T_3, T_4 : T_1 \circ T_2^{-1} \circ T_3 \circ T_4^{-1} \circ T_1^{-1} \circ T_2 \circ T_3^{-1} \circ T_4 = Id \rangle$ be the Fuchsian group where the transformations T_i's are obtained by (5) and (4). The generators of the Fuchsian group $\Gamma \simeq \Gamma_8$ are given by

$$G_i = f^{-1}A_i f,\ i = 1, \cdots, 4,$$

where f is as in (3). Using the software *Mathematica* we obtain the following generators:

$$G_1 = \begin{pmatrix} \frac{2+2\sqrt{2}}{2} + \frac{\sqrt{2}\sqrt[4]{2}}{2} & -\frac{(2+\sqrt{2})\sqrt[4]{2}}{2} \\ -\frac{(2+\sqrt{2})\sqrt[4]{2}}{2} & \frac{2+2\sqrt{2}}{2} - \frac{\sqrt{2}\sqrt[4]{2}}{2} \end{pmatrix}, G_2 = \begin{pmatrix} \frac{2+2\sqrt{2}}{2} + \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} & -\frac{\sqrt{2}\sqrt[4]{2}}{2} \\ -\frac{\sqrt{2}\sqrt[4]{2}}{2} & \frac{2+2\sqrt{2}}{2} - \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} \end{pmatrix},$$
$$G_3 = \begin{pmatrix} \frac{2+2\sqrt{2}}{2} + \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} & \frac{\sqrt{2}\sqrt[4]{2}}{2} \\ \frac{\sqrt{2}\sqrt[4]{2}}{2} & \frac{2+2\sqrt{2}}{2} - \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} \end{pmatrix}, G_4 = \begin{pmatrix} \frac{2+2\sqrt{2}}{2} + \frac{\sqrt{2}\sqrt[4]{2}}{2} & \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} \\ \frac{(2+\sqrt{2})\sqrt[4]{2}}{2} & \frac{2+2\sqrt{2}}{2} - \frac{\sqrt{2}\sqrt[4]{2}}{2} \end{pmatrix}, \tag{7}$$

We have that the generators obtained in (7) are identified via the isomorphism (2) with the following elements of $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $i^2 = \sqrt{2},\ j^2 = -1,\ ij = -ji$.

$$g_1 = \frac{2+2\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{2+\sqrt{2}}{2}ij,\ g_2 = \frac{2+2\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i - \frac{\sqrt{2}}{2}ij,$$
$$g_3 = \frac{2+2\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}ij,\ g_4 = \frac{2+2\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i + \frac{2+\sqrt{2}}{2}ij \tag{8}$$

The only prime ideal that is ramified in $\mathcal{A}$ is the principal ideal $\mathcal{I} = \langle 0, 1 \rangle \subset \mathbb{Z}[\sqrt{2}]$. Then the discriminant of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \langle \sqrt{2} \rangle.$$

Let $R = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. Then

$$\mathcal{O} = \{y_0 + y_1 i + y_2 j + y_3 ij\ :\ y_0, y_1, y_2, y_3 \in \mathbb{Z}[\sqrt{2}]\}$$

is a quaternion order of $\mathcal{A}$ denoted by $\mathcal{O} = (\sqrt{2}, -1)_R$ with $\mathbb{Z}$-basis $\{1, i, j, ij\}$. And

$$\mathcal{D}(\mathcal{O}) = \sqrt{det(Trd(y_i\overline{y_j}))} = \sqrt{32} = 4\sqrt{2}.$$

As $\mathcal{D}(\mathcal{A}) \neq \mathcal{D}(\mathcal{O})$ then the quaternion order $\mathcal{O}$ is not maximal.
Using the software *Magma*, we obtain the order $\mathcal{M} \supset \mathcal{O}$ with R-basis

$$B = \left\{1, i, \frac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j), \frac{1}{2}((\sqrt{2}+1)i + ij)\right\}.$$

First we will show that $\mathcal{M}$ characterized by the basis B is indeed an order. For this, we need to show that $y \in \mathcal{M} \Rightarrow Trd(y), Nrd(y) \in R = \mathbb{Z}[\sqrt{2}]$, and we have that

$$Trd(1) = 2,\ Nrd(1) = 1,\ Trd(i) = 0,\ Nrd(i) = -\sqrt{2},\ Trd(\tfrac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j)) = \sqrt{2} + 1$$
$$Nrd(\tfrac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j)) = 1,\ Trd(\tfrac{1}{2}((\sqrt{2}+1)i + ij)) = 0,\ Nrd(\tfrac{1}{2}((\sqrt{2}+1)i + ij)) = -\sqrt{2} - 1.$$

Furthermore, the discriminant of $\mathcal{M}$ is given by

$$\mathcal{D}(\mathcal{M}) = \sqrt{det(Trd(y_i\overline{y_j}))} = -\sqrt{2}.$$

And so

$$\mathcal{D}(\mathcal{M}) = \mathcal{D}(\mathcal{A}) = \langle \sqrt{2} \rangle.$$

Therefore, $\mathcal{M}$ is a maximal order of $\mathcal{A}$. Now, for associating the elements of the maximal order $\mathcal{M}$ with the elements of the Fuchsian group Γ_8, we have to show that $g_1, g_2, g_3, g_4 \in \mathcal{M}$, where g_1, g_2, g_3, g_4 are as in (8). Indeed, they can be written by linear combination of B

- $a_1 = 1 + \sqrt{2},\ b_1 = 2 + 2\sqrt{2},\ c_1 = 0,\ d_1 = -2 - \sqrt{2} \in \mathbb{Z}[\sqrt{2}] \Rightarrow$
  $a_1 \cdot 1 + b_1 \cdot i + c_1 \cdot \frac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j) + d_1 \cdot \frac{1}{2}((\sqrt{2}+1)i + ij) = g_1$
- $a_2 = 1 + \sqrt{2},\ b_2 = 2 + 2\sqrt{2},\ c_2 = 0,\ d_2 = -\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \Rightarrow$
  $a_2 \cdot 1 + b_2 \cdot i + c_2 \cdot \frac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j) + d_2 \cdot \frac{1}{2}((\sqrt{2}+1)i + ij) = g_2$
- $a_3 = 1 + \sqrt{2},\ b_3 = 0,\ c_3 = 0,\ d_3 = \sqrt{2} \in \mathbb{Z}[\sqrt{2}] \Rightarrow$
  $a_3 \cdot 1 + b_3 \cdot i + c_3 \cdot \frac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j) + d_3 \cdot \frac{1}{2}((\sqrt{2}+1)i + ij) = g_3$
- $a_4 = 1 + \sqrt{2},\ b_4 = -2 - \sqrt{2},\ c_4 = 0,\ d_4 = 2 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}] \Rightarrow$
  $a_4 \cdot 1 + b_4 \cdot i + c_4 \cdot \frac{1}{2}((\sqrt{2}+1) + \sqrt{2}i + j) + d_4 \cdot \frac{1}{2}((\sqrt{2}+1)i + ij) = g_4$

Others Examples

In the same way made to the group Γ_8 we can get maximal orders to the arithmetic Fuchsian groups Γ_g of the tessellation $\{4g, 4g\}$ for other values of g. In the table below, we present some maximal orders for different values of g.

g	θ	$\mathbb{Z}[\theta]$-basis B of $\mathcal{M}$
3	$2\sqrt{3}$	$\{1, i, \frac{1}{2}(1 + (1+\theta)i + j), \frac{1}{2}((1+\theta) + i + k)\}$
4	$\sqrt{2+\sqrt{2}}$	$\{1, i, \frac{1}{2}((\theta^3+\theta^2+1) + \theta^3 i + j), -\frac{1}{2\theta}(2 + (\theta^3+\theta^2-1)i + k)\}$
5	$\frac{\sqrt{10+2\sqrt{5}}}{2}$	$\{1, i, \frac{1}{2}(\theta^3 + j), (-\frac{1}{2}\theta + \frac{1}{10}\theta^3)((\theta^3-2)i + k)\}$
6	$\sqrt{2+\sqrt{3}}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}((\theta^3+\theta+1) + (\theta^3+\theta^2+\theta+1)i + j), \frac{1}{2}((\theta^3+\theta^2+\theta+1) + (\theta^3+\theta+1)i + k)\}$
8	$\sqrt{2+\sqrt{2+\sqrt{2}}}$	$\{1, i, \frac{1}{2}((\theta^7+\theta^6+\theta^4+1) + \theta^7 i + j), -\frac{1}{2\theta}(2 + (\theta^7+\theta^6+\theta^4-1)i + k)\}$
10	$\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}(\theta^6 + j), \frac{1}{2}(\theta^6 i + k)\}$
12	$\sqrt{2+\sqrt{2+\sqrt{3}}}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}((\theta^7+\theta^6+\theta^5+\theta^3+\theta^2+\theta+1) + (\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1)i + j),$ $\frac{1}{2}((\theta^7+\theta^6+\theta^5+\theta^3+\theta^2+\theta+1) + (\theta^7+\theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1)i + k)\}$
15	$\frac{\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}}{2}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}(\theta^5+\theta^3+\theta + j), \frac{1}{2}((\theta^5+\theta^3+\theta)i + k)\}$
20	$\sqrt{2+\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}(\theta^{12} + j), \frac{1}{2}(\theta^{12} i + k)\}$
30	$\sqrt{2+\frac{\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}}{2}}$	$\{1, -\frac{1}{\theta}i, \frac{1}{2}(\theta^{10}+\theta^6+\theta^2 + j), \frac{1}{2}((\theta^{10}+\theta^6+\theta^2)i + k)\}$

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