Index Coding

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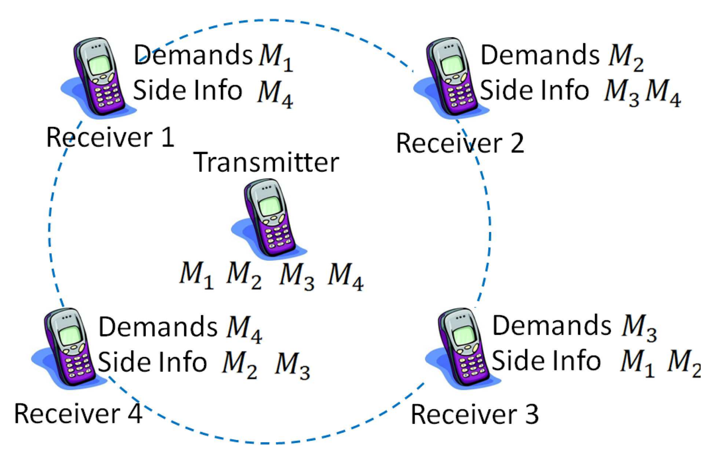
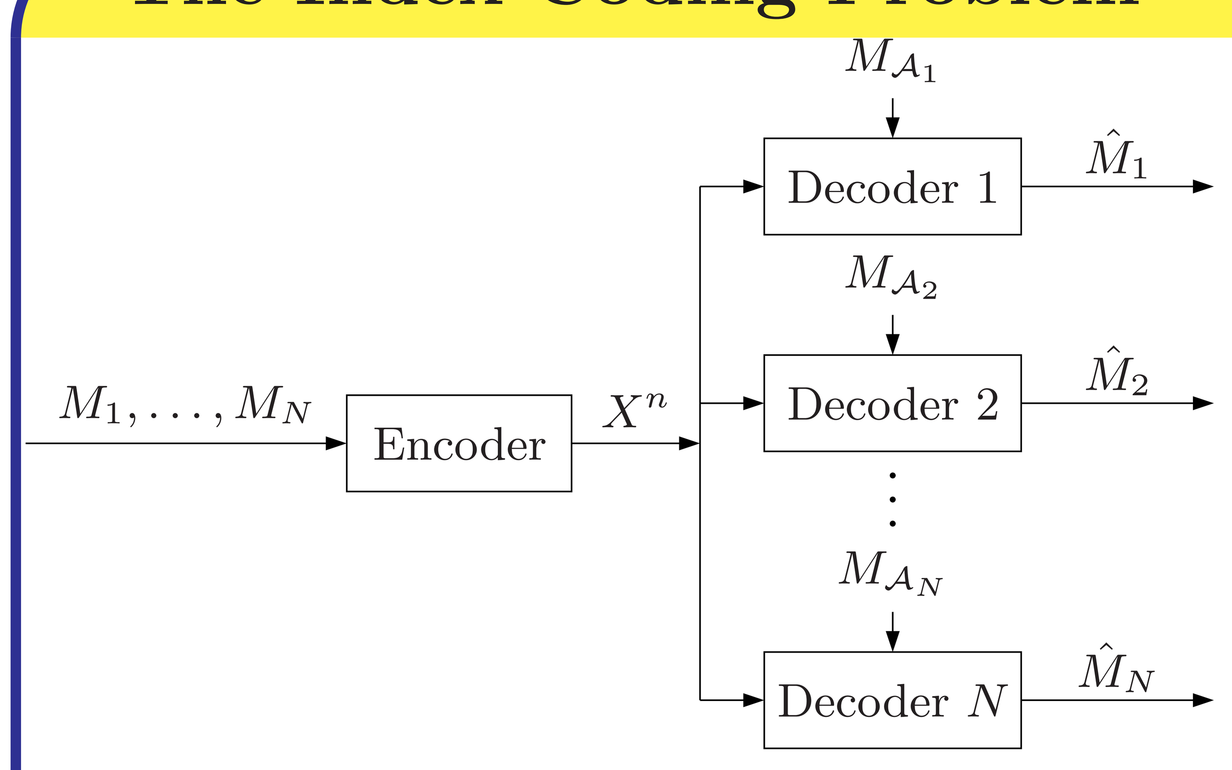
Example



- Naive strategy:
  - Send one message at a time (requires four transmissions)
- Coding:
  - Send coded messages $M_1 + M_2 + M_3$ and $M_1 + M_4$ (requires two transmissions)
- Coding can decrease the number of transmissions

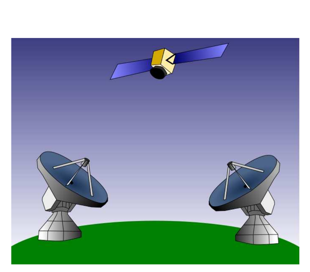
The Index Coding Problem



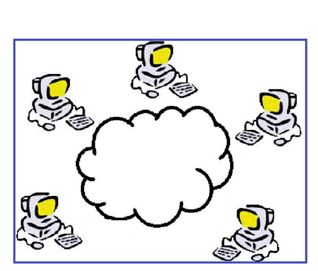
- $M_j \in [1 : 2^{nR_j}]$, $j \in [1 : N]$
- Side information at receiver j: $\mathcal{A}_j$
- Compact representation: $(j|\mathcal{A}_j)$, $j \in [1 : N]$

Applications

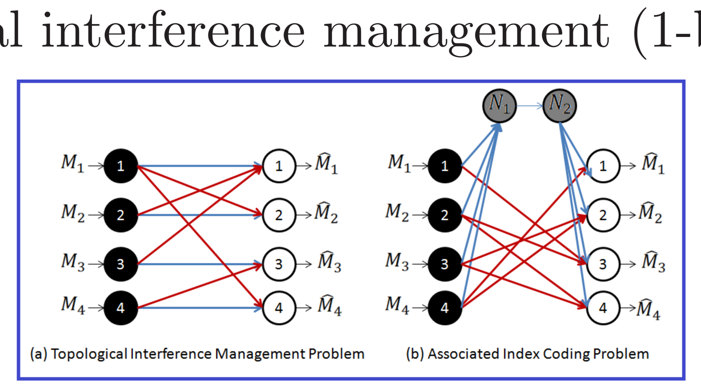
Satellite Communication



Content distribution



Topological interference management (1-bit CSIT)



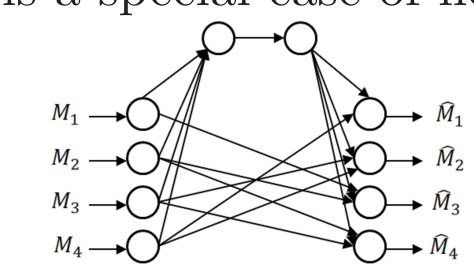
- The capacity of the interference management problem is upper bounded by the capacity of the corresponding index coding problem [Jafar2013].

Prior capacity lower bounds

- Combinatorial approaches
  - Chromatic number [Bar-Yossef et al. 2006]
  - Fractional chromatic number [Alon et al. 2008]
  - Local chromatic number [Shanmugam et al. 2013]
- Interference alignment approaches
  - One-to-one alignment [Maleki et al. 2012]
  - Subspace alignment [Maleki et al. 2012]

Index Coding vs. Network Coding

- Index coding is a special case of network coding



- Index coding and network coding problems are equivalent [Effros–El-Rouayheb–Langberg2012]

Flat Coding

- Codebook generation: For each $(m_1, \ldots, m_N)$, generate a random binary sequence $x^n(m_1, \ldots, m_N)$
- Encoding: To send $(m_1, \ldots, m_N)$, transmit $x^n(m_1, \ldots, m_N)$
- Decoding: Each receiver recovers all of the messages

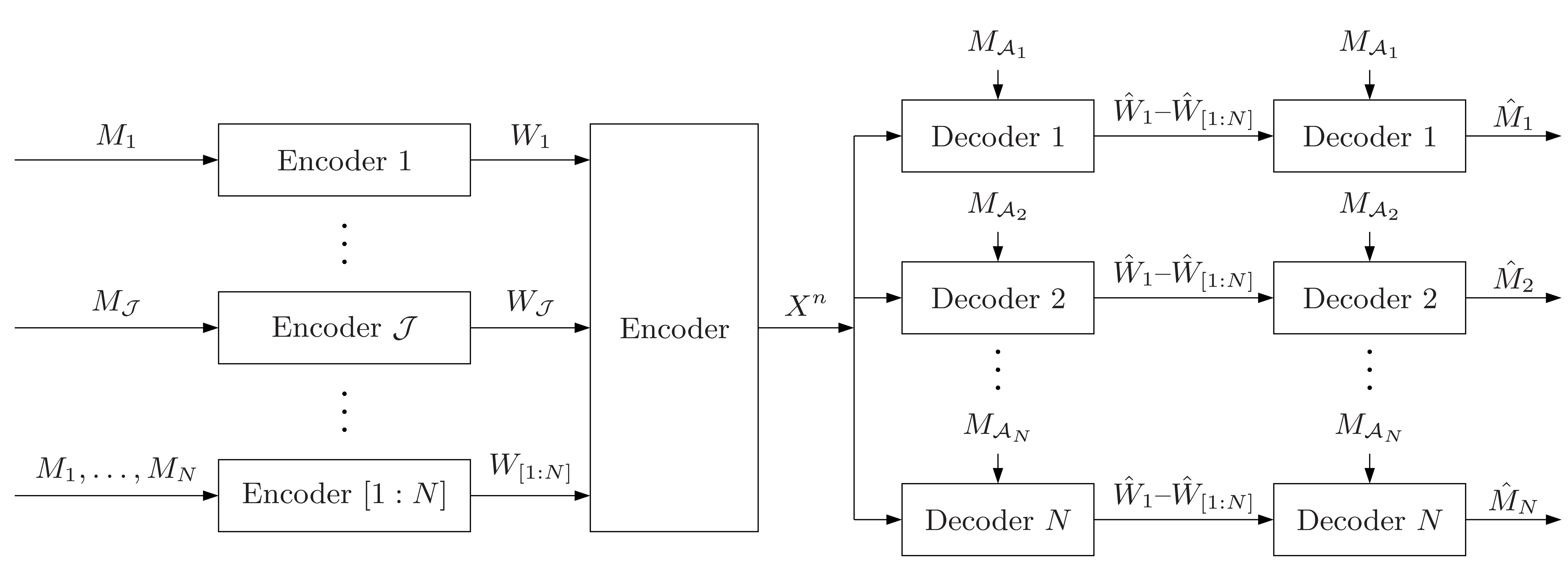
Flat Coding Inner Bound

$$\sum_{k \notin \mathcal{A}_j} R_k < 1 \quad \text{for all } j \in [1 : N]$$

Example: $(1|4)$, $(2|3, 4)$, $(3|1, 2)$, $(4|2, 3)$

- Receiver 1 finds the unique $(\hat{m}_1, \hat{m}_2, \hat{m}_3, m_4)$ such that $x^n(\hat{m}_1, \hat{m}_2, \hat{m}_3, m_4) = x^n$
- Number of wrong tuples: $2^{nR_1}2^{nR_2}2^{nR_3} - 1$
- Probability of two identical codewords: $1/2^n$
- By union bound $P_e < 2^{nR_1}2^{nR_2}2^{nR_3}\frac{1}{2^n}$
- $P_e \to 0$ if $R_1 + R_2 + R_3 < 1$
- similarly we obtain
  $R_1 + R_2 < 1$ (inactive)
  $R_3 + R_4 < 1$
  $R_1 + R_4 < 1$
- Flat coding is not optimal ($R_{sym} = 1/3$)

Composite Coding



Composite Coding Inner Bound

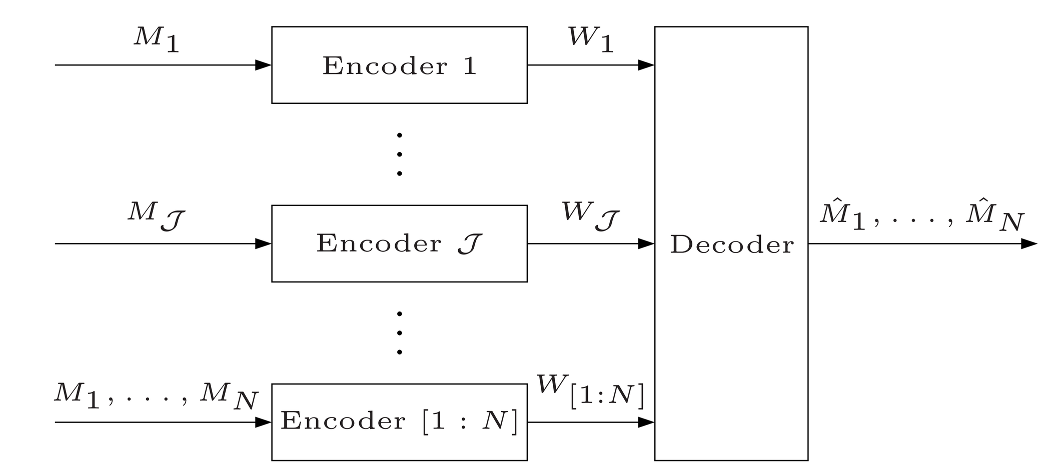
$$(R_1, \ldots, R_N) \in \bigcap_{j \in [1:N]} \bigcup_{\mathcal{K} \subseteq [1:N] : j \in \mathcal{K}} \mathscr{R}(\mathcal{K}|\mathcal{A}_j)$$

for some $(S_{\mathcal{J}} : \mathcal{J} \subseteq [1 : N])$ such that $S_{\mathcal{J}} \geq 0$ for all $\mathcal{J} \subseteq [1 : N]$ and $\sum_{\mathcal{J} : \mathcal{J} \not\subseteq \mathcal{A}_j} S_{\mathcal{J}} < 1$ for all $j \in [1 : N]$

- Polymatroidal region $\mathscr{R}(\mathcal{K}|\mathcal{A})$: $\sum_{j \in \mathcal{J}} R_j < \sum_{\mathcal{J}' \subseteq \mathcal{K} \cup \mathcal{A} : \mathcal{J}' \cap \mathcal{J} \neq \emptyset} S_{\mathcal{J}'}$ for all $\mathcal{J} \subseteq \mathcal{K} \setminus \mathcal{A}$

Dual Index Coding

- Goal: Send a message tuple $(M_1, \ldots, M_N)$
- A set of $(2^N - 1)$ senders
- Each sender encodes a subtuple $M_{\mathcal{J}}$ into an index $W_{\mathcal{J}} \in [1 : 2^{nS_{\mathcal{J}}}]$
- What is the capacity region (as a function of $S_{\mathcal{J}}$)?



Capacity Region of the Dual Index Coding

$$\sum_{j \in \mathcal{J}} R_j \leq \sum_{\mathcal{J}' \subseteq [1:N] : \mathcal{J}' \cap \mathcal{J} \neq \emptyset} S_{\mathcal{J}'}$$

for all $\mathcal{J} \subseteq [1 : N]$

- Special case of the general multiple access channel with correlated messages [Han1979] and [Slepian–Wolf1973]
- The capacity region can be achieved by random coding and simultaneous decoding

Outer Bound

- Submodularity outer bound [Blasiak et al. 2011],[Dougherty et al. 2011] is based on
  - Fano's inequality
  - Submodularity of entropy
- Not tight in general [Sun-Jafar 2013]

Results

Composite coding

✓ Optimal up to $N = 5$

✓ Achieves the optimal symmetric rate of the class of symmetric index coding problems with neighboring antidotes
  - Every receiver knows D messages to the left and U messages to the right

✓ Better than fractional local graph coloring

✓ Better than one-to-one interference alignment

✗ Rate region is not easy to evaluate (1540944 problems with $N = 6$)

✗ Sometimes worse than subspace interference alignment