Channel Capacity under Sub-Nyquist Nonuniform Sampling

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Background

• Information Theory Meets Sampling Theory

  ![Diagram of information theory and sampling theory](image)

  - How to jointly optimize the input distribution and sampling methods?

  • What is Sampled Channel Capacity [ChenEldarGoldsmith'2011]
    – For a given sampling system:
      \[ x(t) \xrightarrow{H(t)} \hat{y}(t) \xrightarrow{\text{Sampler}} y[n] \]
      – For a large class of sampling systems
      \[ x(t) \xrightarrow{\text{Sampler}} \hat{y}(t) \]
      Joint Optimization (Input and Sampling Methods)

Motivation

• Consider a bank of filters each followed by a uniform sampler...

![Diagram of a bank of filters and uniform samplers](image)

  – The sampled capacity is nonmonotonic in the sampling rate

Questions

• Will more general nonuniform sampling methods improve capacity?
• Which sampling systems can maximize capacity for a given sampling rate?
• What is the gap between sub-Nyquist sampled capacity and analog capacity?

Problem Formulation

• Sampling Rate
  - Beurling Density: \( f_s = \lim_{T \to \infty} \inf_t T \left[ N / (t_n f_s + T) \right] \)

  ![Diagram of sampling rate](image)

  - Time-preserving Preprocessing System: A system that preserves the time scales
    – Counterexample: \( T(x(t)) = x(2t) \)

  • Sampled Channel Capacity (Perfect Channel State Information at Both Sides)
    – For a given system \( P \):
      \[ C^P(f_s) = \lim_{T \to \infty} \inf_{t \in B_0} \sup_{\mu(B_0) = f_s} \frac{1}{T} \int \left[ x([-T,T], \{\hat{y}(t)\}_{[-T,T]}) \right] \]
    – For the class of time-preserving systems: \( C(f_s) = \lim \sup_{T} C^P(f_s) \)

Converse: (Main Result)

• Theorem 1. Consider any time-preserving sampling system with rate \( f_s \).
  Suppose that there exists a frequency set \( B_{\text{lin}} \) that satisfies \( \mu(B_{\text{lin}}) = f_s \) and
  \[ \int_{f \in B_{\text{lin}}} |H(f)|^2 \omega_0(f) \]
  \[ \cdot \sup_{B \in \mu(B) = f_s} \int_{f \in B} |H(f)|^2 \omega_0(f) \]
  Then the sampled channel capacity can be upper bounded by
  \[ C_s(f_s, P) = \int_{f \in B_{\text{lin}}} \frac{1}{2} \log \left( \frac{|H(f)|^2 \omega_0(f)}{S_0(f)} \right) df, \]

Achievability

• The upper bound in Theorem 1 can be achieved by sampling via a filter bank:

  ![Diagram of a filter bank](image)

  or be approached by sampling via a single branch of modulation and filtering:

Implications

• The Optimal Sampling Method:
  – extracts out a frequency set with the highest SNR
  – suppresses aliasing
  – results in a capacity monotonic in the sampling rate
  – robust to mild permutation of the sampling grid

• Irregular nonuniform sampling grid does not improve capacity.

• When the sampling rate is increased, the adjustment of the sampling hardware for filter-bank sampling is incremental.

The Way Ahead

• If the CSI is not perfectly known or if the channel state can be varying:
  – Alias-suppressing sampling is not necessarily optimal.
  – May need to scramble spectral contents.
  – May need different objective metrics (e.g. minimaxity).

• Decoding-constrained information theory:
  – Sampling systems can be viewed as part of the decoding method.
  – How to find the capacity-achieving input and decoding strategy if the decoding strategy needs to be picked from a given set (possibly infinitely many choices)