Lookahead, Estimation and Information in the Gaussian Channel

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Introduction

- **The Setup:** Mean squared estimation of a signal corrupted by additive white Gaussian noise.
- Continuous-time Gaussian Channel:

$$dY_t = \sqrt{\gamma} X_t dt + dW_t, \quad (1)$$

where $\mathbf{X} = \{X_t : -\infty < t < \infty\}$ denotes the channel input process, and $\mathbf{Y}$ is the output process at Signal-to-Noise ratio γ.
- $\mathbf{W}$. is a standard Brownian Motion independent of $\mathbf{X}$.

Background

Let $\mathbf{X}$ be a stationary process. Define:

- Mutual Information rate

$$I(\gamma) = \lim_{T \to \infty} \frac{I(X_0^T; Y_0^T)}{T}$$

- Non-causal mean squared error

$$\mathrm{mmse}(\gamma) = \mathbf{E}\left[(X_0 - \mathbf{E}[X_0 | Y_{-\infty}^{+\infty}])^2\right]$$

- Causal mean squared error

$$\mathrm{cmmse}(\gamma) = \mathbf{E}\left[(X_0 - \mathbf{E}[X_0 | Y_{-\infty}^{0}])^2\right]$$

Triangular Relationship

From [1] and [2], we know that for all snr > 0,

$$\frac{2I(\mathrm{snr})}{\mathrm{snr}} = \mathrm{cmmse}(\mathrm{snr}) = \frac{1}{\mathrm{snr}} \int_0^{\mathrm{snr}} \mathrm{mmse}(\gamma)\, d\gamma. \quad (2)$$

- The mutual information rate function **completely characterizes** the causal and smoothing errors as functions of snr.

MMSE with Lookahead

- In this work, we investigate the role of finite lookahead, in information and estimation under mean squared loss.
- The mean squared error with finite lookahead d is defined as

$$\mathrm{lmmse}(d, \gamma) = \mathbf{E}\left[(X_0 - \mathbf{E}[X_0 | Y_{-\infty}^{d}])^2\right], \quad (3)$$

- Note that $d = 0$ and $d = \infty$ in (3) yield the causal and non-causal errors respectively.

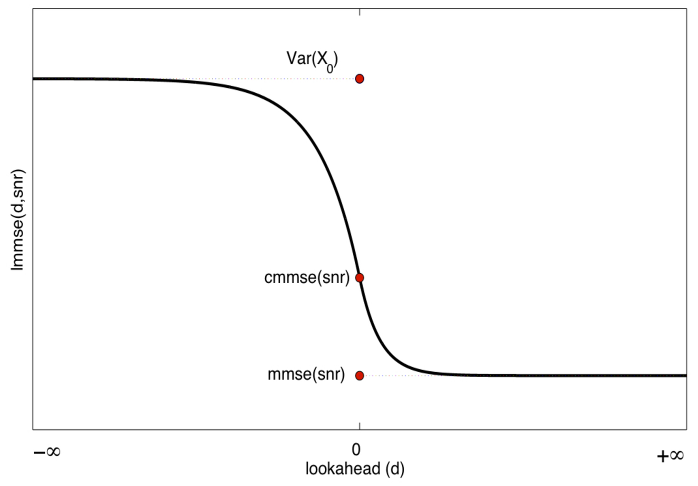

Figure: Characteristic behavior of the minimum mean squared error with lookahead for any input process, corrupted by the Gaussian channel.

Note:
The mutual information rate as a function of SNR determines the three points corresponding to the value at $d = 0$, and the asymptotes at $\pm\infty$ in the above curve.

Question

Can the mutual information rate function $\mathbf{I}(\cdot)$, determine the MMSE with lookahead for finite, non-zero lookahead $\mathbf{d}$?

Theorem

For any finite $d < 0$ there exist stationary continuous-time processes which have the same mutual information rate $I(snr)$ for all snr, but have different minimum mean squared errors with (negative) lookahead d.

Main Proof Ideas

- By Duncan's result [1], the causal and anti-causal errors as functions of snr are the same (due to the mutual information acting as a bridge, which is invariant to the direction of time). I.e.,

$$\mathrm{Var}(X_0 | Y_{-\infty}^{0}) = \mathrm{Var}(X_0 | Y_0^{\infty}). \quad (4)$$

- We present an explicit construction of a process for which

$$\mathrm{Var}(X_0 | Y_{-\infty}^{d}) \neq \mathrm{Var}(X_0 | Y_{-d}^{\infty}) \quad (5)$$

for some values of d. Note that the left and right sides of (5) are the MMSE's with lookahead d associated with the original process, and its **time reversed version**, respectively. Thus, mutual information alone does not characterize these objects.

Process Construction

- Let $\tilde{\mathbf{X}} = \{\tilde{X}_i\}_{i=-\infty}^{+\infty}$ be a discrete time Markov Chain. Define a piecewise constant continuous-time process X_t such that

$$X_t \equiv \tilde{X}_i \quad t \in (i-1, i] \quad (6)$$

- We now apply a random shift $\Delta \sim U[0, 1]$ to the $\{X_t\}$ process to make it stationary.The resulting process $\mathbf{X}$ is observed through the Gaussian channel.
- Similarly, we construct the process $\mathbf{X}^{(R)}$, which is generated from the discrete-time process $\tilde{\mathbf{X}}^{(R)}$ (time-reversed version of $\tilde{\mathbf{X}}$) using the same procedure.
- We compare the minimum mean square errors with finite lookahead for the processes $\mathbf{X}$ and $\mathbf{X}^{(R)}$, for a certain underlying discrete-time Markov process.

Simulations using Markov Chain Monte Carlo

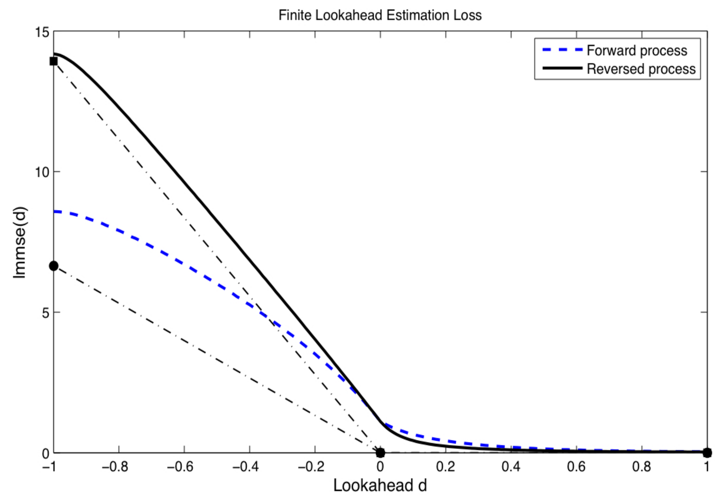

Figure: Comparison of the Estimation Loss with finite lookahead for the Forward and Reverse processes, $\mathbf{X}$ and $\mathbf{X}^{(R)}$ respectively at SNR=1. For reference, the (analytical) infinite SNR curves (dashed) are also shown. Note that the curves meet at $d = 0$, in agreement with Duncan's result.

Discussion

Thus, $\mathbf{X}$ and $\mathbf{X}^{(R)}$ have the same mutual information rate function, but different MMSE's with lookahead for both positive and negative lookahead.

Lookahead vs. SNR: A Generalized Observation Model

Letting Y_t denote the channel output, we describe the channel as:

$$dY_t = \begin{cases} \sqrt{\mathrm{snr}} X_t\, dt + dW_t & t \leq 0 \\ \sqrt{\gamma} X_t\, dt + dW_t & t > 0 \end{cases} \quad (7)$$

where, as usual, W. is a standard Brownian motion independent of $\mathbf{X}$. Note that for $\gamma = \mathrm{snr}$, we recover the usual time-invariant Gaussian channel.

Definition:
Letting $d, l \geq 0$, we define the finite lookahead estimation error at time d with lookahead l as

$$f(\mathrm{snr}, \gamma, d, l) = \mathrm{Var}(X_d | Y_{-\infty}^{l+d}). \quad (8)$$

Theorem

Let X_t be any finite variance continuous time stationary process which is corrupted by the Gaussian channel in (7). Let f be as defined in (8). For $snr > 0$ and $T > 0$, we have

$$cmmse(snr) = \frac{1}{T \cdot snr} \int_0^{snr} \int_0^{T} f(snr, \gamma, t, T - t)\, dt\, d\gamma \quad (9)$$

Discussion

The above theorem presents a trade-off between lookahead and signal-to-noise ratio of the channel, as a double integral, with the causal MMSE emerging as a quantity that is conserved under this operation.

Additional Results

In addition to detailed proofs and discussions of the above results, in [3], we present some new results relating to the role of lookahead in information-estimation.

- We explicitly characterize the MMSE with lookahead for the class of stationary Gauss-Markov processes, and their mixtures.
- We introduce the notion of Information Utility of small lookahead, and show that under basic regularity conditions on the input process, this quantity is characterized by the causal minimum mean squared error.

References

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