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The Capacity of a Class of Relay Channels with State

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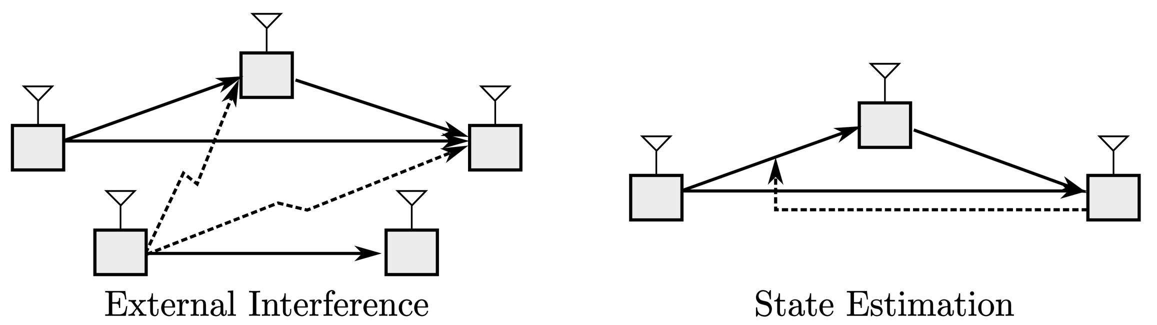
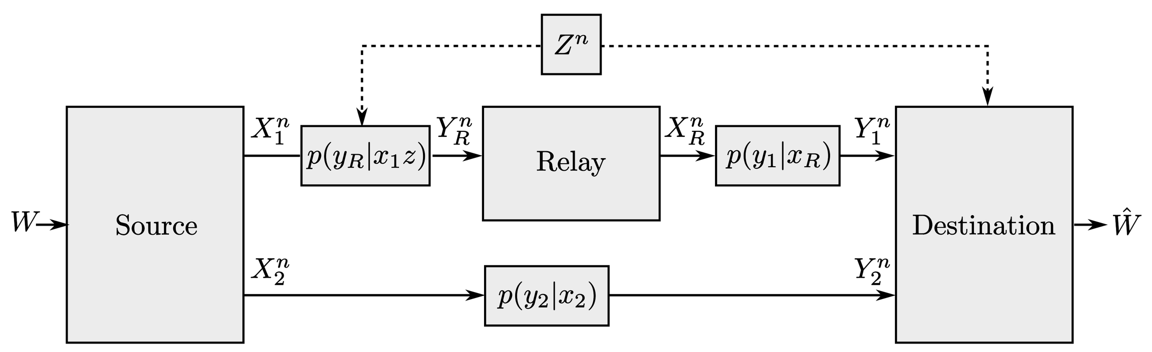
Introduction

- Relay channel was introduced in 1971 by van der Meulen, but the capacity is still open in general.
- Capacity is known for some special cases.



- Noise can be correlated due to an external common interference.
- Knowledge about the state can be available at the destination.
  - Self-interference.
  - State estimation.

System Model



- $W \in \{1, ..., M\}$ message to be transmitted.
- An (M, n) code for this channel consists of
  - Encoding function at source: $f : \{1, ..., M\} \to \mathcal{X}_1^n \times \mathcal{X}_2^n$,
  - Set of encoding functions at relay: $X_{Ri} = f_r(Y_{R1}, ..., Y_{R(i-1)}) \quad i = 1, ..., n$.
  - Decoding function at destination: $g : \mathcal{Y}_1^n \times \mathcal{Y}_2^n \times \mathcal{Z}^n \to \{1, ..., M\}$.
- Probability of error: $P_e = \frac{1}{M}\sum_w Pr\{g(Y_1^n, Y_2^n, Z^n) \neq w | W = w\}$.
- R achievable if exists a sequence of $(2^{nR}, n)$ codes s.t. $P_e \to 0$ as $n \to \infty$.
- The capacity $\mathcal{C}$ is the supremum of the set of achievable rates.

Research question

- **Q1: Can we find a single-letter expression for the capacity of this special relay channel model?**
- Known transmission strategies
  - **Partial Decode and Forward (pDF):** Relay decodes part of the message.

$$R_{DF} = \sup_{p(x,x_1)p(y,y_1|x,x_1)} \min\{I(X; Y|X_1), I(X, X_1; Y)\}.$$

  - Optimal for:
    - Physically degraded relay channel, inversely degraded relay channel [1].
    - Orthogonal relay channel [2].
    - Semi-deterministic relay channel.
  - **Compress and Forward (CF)**: Relay compresses its received signal.

$$R_{CF} = \sup_{p(x_1)p(x_2)p(y,y_1|x_1,x_2)p(\hat{y}_1|y_1x_2)} I(X_1; Y\hat{Y}_1|X_2)$$
$$\text{s.t. } I(\hat{Y}_1; Y_1|Y, X_1) \leq I(X_1; Y|V).$$

  - Optimal for:
    - A class of deterministic relay channels [3].
    - A class of modulo-sum relay channels [4].
  - **Partial Decode Compress and Forward (PDCF)**: The relay decodes part of the message and compresses the remainder.

$$R_{PDCF} = \sup \min\{I(X; Y, \hat{Y}_1|X_1, U) + I(U; Y_1|X_1, V), I(X, X_1; Y) - I(\hat{Y}_1; Y_1|X, X_1, U, Y)\},$$
$$\text{s.t. } I(\hat{Y}_1; Y_1|Y, X_1, U) \leq I(X_1; Y|V),$$
$$\text{over } p(v)p(u|v)p(x|u)p(x_1|v)p(y, y_1|x, x_1)p(\hat{y}_1|x_1, y_1, u).$$

  - Optimal for:
    - A class of diamond relay channels [5].
    - Up to now, not shown to be capacity achieving for any single relay channel model.
- **Q2: Is any of these schemes capacity-achieving in our setting?**

Main Result

Theorem (Capacity of the Orthogonal Relay with Channel State Knowledge)

The capacity of the orthogonal relay channel with state, $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_R, p(z)p(y_R|x_1z)p(y_1|x_R)p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_R)$*, is given by*

$$\mathcal{C} = \sup_{\mathcal{P}} R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R|UZ),$$
$$\text{s.t. } R_0 \geq I(U; Y_R) + I(Y_R; \hat{Y}_R|UZ).$$

where

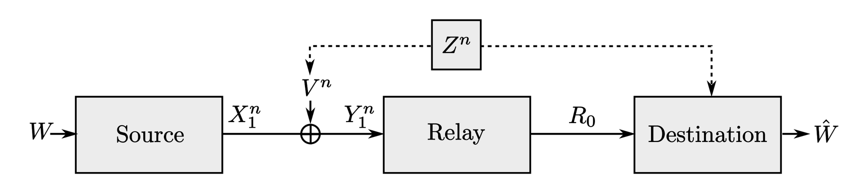
$$\mathcal{P} \triangleq \{p(ux_1zy_R\hat{y}_R) : p(u, x_1)p(z)p(y_R|x_1z)p(\hat{y}_R|y_Ru)\}.$$

and

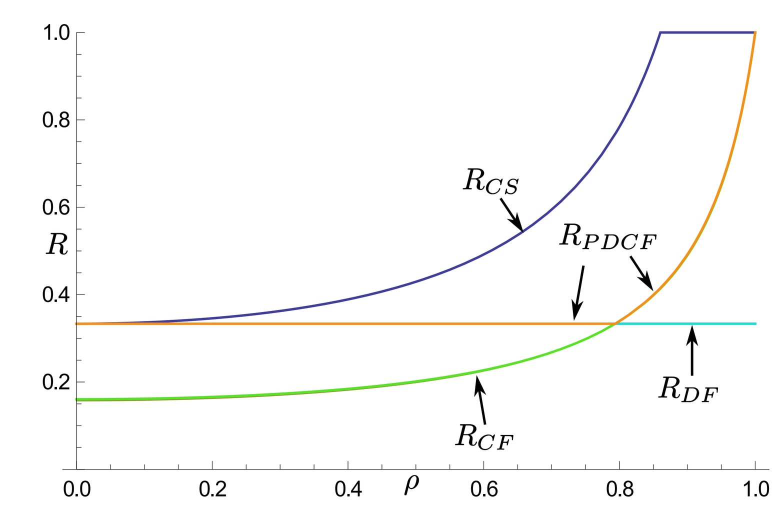
$$R_0 \triangleq \max_{p(x_R)} I(X_R; Y_1), \qquad R_1 \triangleq \max_{p(x_2)} I(X_2; Y_2),$$

Example 1: Multihop Gaussian Relay Channel

- Let $R_1 = 0$, (V, Z) bivariate Gaussian with correlation coefficient ρ and $Y_R = X_1 + V$.



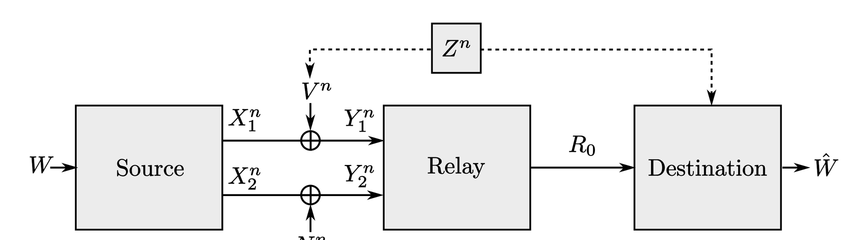
- Evaluating with Gaussian auxiliary r.v.'s (potentially suboptimal)



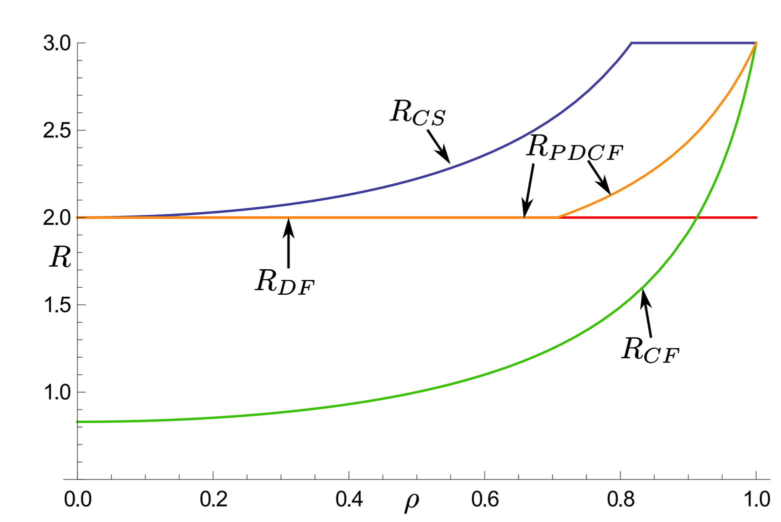
- PDCF reduces to the best of DF and CF!

Example 2: Orthogonal Multihop Gaussian Relay Channel

- Add a parallel channel with $N \sim (0, 1)$ independent of (V, Z).



- PDCF becomes better than either DF or CF! (with Gaussian r.v.'s)



Outline of the Proof

- **Converse**:
  - Single letter upper bound

$$R_{up} = \sup_{\mathcal{P}} \min\{R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R|UZ), R_0 + R_1 - I(\hat{Y}_R; Y_R|X_1UZ)\}.$$

  - Equivalent expression for R_{up}

$$R_{up} = \sup_{\mathcal{P}} R_1 + I(U; Y_R) + I(X_1; \hat{Y}_R|UZ)$$
$$\text{s.t. } R_0 \geq I(U; Y_R) + I(\hat{Y}_R; Y_R|UZ).$$

- **Achievability**
  - PDCF evaluated with
    - $X^n = (X_1^n, X_2^n)$, $Y^n = (Y_1^n, Y_2^n, Z^n)$, $V = \emptyset$.
    - X_R^n and X_1^n independent of the rest of variables and $p(x_R^*)$ and $p(x_1^*)$ are capacity achieving for corresponding channels.

Is $\mathcal{C}$ Below the Cut-Set Bound?

- Cut-Set Bound

$$R_{CS} = \sup_{p(x_1)p(z)p(y_R|x_1z)} \min\{R_0 + R_1, R_1 + I(X_1; Y_R|Z)\}.$$

- When $\mathcal{C} = R_{CS}$?
  - $R_0 \leq I(X_1; Y_R)$ (DF optimal).
  - Y_R independent of Z (DF optimal).
  - $Y_R = f(X_1, Z)$ (CF optimal).
- For Example 1, R is below the cut set bound:

$$R_{CS} = \min\left\{R_0, R_1 + \log\left(1 + \frac{P}{1 - \rho^2}\right)\right\}.$$

  - Assume $R_{CS} = R_0$.
  - Necessary condition for $\mathcal{C} = R_{CS}$:

$$I(V; \hat{Y}_R|UZX_1) = 0$$

  - This condition can only be satisfied if $(N\hat{Y}_RUZX_1)$ are jointly Gaussian.
  - Gaussian r.v.'s do not meet the cut-set bound in general (See Example 1).

References

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Acknowledgments

This work is supported in part by a Marie Curie grant funded by European Union's Seventh Framework Programme (FP7). It has received further support from the Catalan Government under grant 2009 SGR 891 and the Spanish Ministry of Science and Innovation (FPU grant AP2009-5007). This template is modified from one of the templates given here
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